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MATHEMATICS II

BCA, Second Semester, Tribhuvan University

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PREFACE

It is my great pleasure to present the book entitled '**Mathematics II**' in front of you all. This book has been designed to supply the theoretical, numerical and computer skills to the students in regard to Mathematics. This book is based on the curriculum of BCA, Second Semester, CAMT 154, TU. It is also a reference material for the students of other programs.

The book is designed under 6 units. Every unit and sub-unit is based on the syllabus of BCA, Second Semester, TU. I hope this book will provide the adequate materials to enhance the study of BCA students. In order to facilitate the students, sufficient solved examples are included. For the practice materials, numbers of exercises are included. At the end of the book, the learning process of Mathematical Software Mathematica and MATLAB are introduced. I hope this will help the students to solve specified problems of the course with the help of Mathematica and MATLAB.

I do not claim on my own originality on the matter of the text nor on its sufficiency. Most of the definitions, theorems, examples and figures are taken from the book Calculus and Analytical Geometry, 9th Edition by G.B. Thomas and R.L. Finney. I am heartily obliged to the authors of this book and to those who are mentioned in the bibliography. I am really indebted to all of them. If due recognition for the authorship of any material is lacking, I extend apology to them.

I am grateful to my seniors, friends and colleagues who encouraged and inspired me to write this book. I warmly acknowledge my wife Asmita Panthi and other family members for their patience and support while writing this book. I would like to acknowledge and thank Mr. Manoj Kumar Sharma, Managing Director of Asmita Books Publisher and Distributors (P.) Ltd. Also, I am thankful to Mr. Bijaya Shrestha for his support in typing and designing the book.

Finally, I would like to take the responsibility for remaining errors, and I would be indebted if they are pointed out to me. Constructive criticism and suggestions for the improvement of the book, either from the teachers or students, will be highly appreciated.

Jhavi Lal Ghimire (9851135564)

2019

PREFACE TO THE THIRD REVISED EDITION

I am pleased to present the third revised edition of the book 'Mathematics II' BCA 154 for second semester of Tribhuvan University. This edition is made according to the new syllabus 2025 of BCA, TU. Several improvements in the book have been made to enhance the clarity and understanding. New topics such as partial derivatives, partial differential equations, Taylor's theorem, Maclaurin theorem, rate measures etc have been added. To make the learning more effective, very short answer questions have been kept at the end of each exercise.

The previous editions were highly appreciated by many teachers as well as students and I am truly thankful for the support and feedback. I believe that this revised edition will also serve as a helpful resource materials for students, teachers and anyone interested in mathematics.

Jhavi Lal Ghimire
2026

CURRICULUM

MATHEMATICS II

Full Marks: 60 x 50 = 3000
Pass Marks: 24 x 50 = 1200
Credit Hours: 3

Course Title: Mathematics II
Course Code: BCA 154
Nature of Course: Theory/ Lab
Year/Semester: I/II

COURSE DESCRIPTION

The course covers the concepts of Limit and Continuity, Derivatives of Algebraic, Trigonometric, Exponential, and Logarithmic functions, and their applications. It also covers Integration and Applications, Volume and Surface Integral of some selected functions, and Numerical Methods using Trapezoidal and Simpson's Rule. Ordinary and Partial Differential Equations, their solutions and uses are also discussed in this course. Other contents of the course include Problems like Simplex Method for two variables, and Gauss-Seidel, Gauss Elimination, LU Inversion, Bisection, and Newton-Raphson Method for solving the linear equations.

COURSE OBJECTIVES

The objective of this course is to enable students to

- understand the concept of limit and continuity of functions and their derivatives, derivative,
- differentiate different types of functions, geometrical meaning of the derivative and its applications to real-world problems,
- integrate functions, understand their meaning and applications, including the volume integrals,
- solve ordinary and partial differential equation and their connections to real-world problems,
- calculate the optimization problems, describe and interpret the practical applications.

COURSE CONTENTS

Unit 1: Limit and Continuity

- 1.1 Definition of limit including epsilon-delta condition, right and left hand limit and infinity
- 1.2 Algebraic properties of limit
- 1.3 Definition and conditions of continuity and discontinuity
- 1.4 Continuity of algebraic, Trigonometric, and exponential functions and their examples.

Unit 2: Derivatives

- 2.1 Definition and geometrical meaning of derivatives,
- 2.2 First principle (or by definition) method to differentiate algebraic, trigonometric, and logarithmic functions,
- 2.3 Rules of derivatives (sum, product, power, chain, and quotient rules),
- 2.4 Derivatives of inverse circular, hyperbolic functions,
- 2.5 Higher order derivatives,
- 2.6 Relation between derivative and continuity
- 2.7 Definition and examples of partial derivatives

Unit 3: Applications

- 3.1 Increasing and decreasing functions
- 3.2 Equation of tangent and normal

- 3.3 L'Hospital's rule,
3.4 Angle between two lines,
3.5 Maxima and minima, absolute maxima and minima, concavity, stationary points and points of inflection,
3.6 Statement and geometrical interpretation of Rolle's theorem, Cauchy Mean-value theorem and Generalized Mean-value theorem,
3.7 Taylor's theorem, Maclaurin theorem (without proof) and its use in expansion of some simple functions,
3.8 Applications of derivatives in Economics,
3.9 Rate measures

Unit 4: Anti-derivative and its Applications 8 Hrs.

- 4.1 Definition and geometrical meaning of integration,
4.2 Basic integration formulas for algebraic, trigonometric, exponential, and logarithmic functions,
4.3 Trigonometric substitution for basic functions, Integration by parts (Product rule for integration),
4.4 Partial fractions,
4.5 Improper integral,
4.6 Definite integral in terms of Riemann sum, and fundamental theorem of integral calculus
4.7 Applications of definite integral (Area under curve, area between curves, Quadrature & rectification)
4.8 Surface and volume integrals,
4.9 Trapezoidal and Simpson's Rule for numerical integration

Unit 5: Differential Equations 8 Hrs.

- 5.1 Definition, order, and degree of differential equations,
5.2 Differential equations of first order and first degree,
5.3 Variables separable, homogeneous, exact, and linear differential equations,
5.4 Reducible to linear form,
5.5 Partial differential equations with some basic examples

Unit 6: Computational Methods 10 Hrs.

- 6.1 Linear programming problems,
6.2 Linear inequalities in two variables and their graphical solutions,
6.3 Simplex Method (up to 3 variables), Duality problems
6.4 Matrix inversion method,
6.5 Gauss Elimination, Gauss-Seidel method,
6.6 Bisection method and Newton-Raphson Method for non-linear equations

LABORATORY WORKS

Students are expected to use Python, MATLAB, or Mathematica to solve the numerical problems of each unit and compare the solution with that of pen and paper.

Examination Scheme				
Internal Assessment		External Assessment		
Theory	Practical	Theory	Practical	
20	20 (3hrs)	60 (3 Hrs)	---	100

REQUIRED READING

- Bajracharya, B. C. (2025). *Basic Mathematic*, Sukunda Pustak Bhawan,
Budnick, F. S. (2017). *Applied Mathematics for Business Economics and Social Sciences*. McGraw-Hill,
Chand, K. B., Sapkota, B. P. (2022). *Principles of Mathematical Analysis*. Pinnacle Publication.
Lay, D. C. (2003). *Linear algebra and its applications*. Pearson Education India.
Stewart, J., Clegg, D., Watson, S. (2020). *Calculus*. Cengage|webassign.
Thomas G. B., Finney, R. L. (1995). *Calculus and Analytical Geometry*. Narosa Publishing House. (Textbook)

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Limit and Continuity

UNIT

1

1.1 Introduction

Limit is the fundamental concept of calculus. In real sense, calculus is the study of limits. It was developed independently by Sir Isaac Newton (1642-1727) and Gottfried Wilhelm Leibnitz (1646-1716). Intuitively, limit is a value to which a function approaches as the variable approaches a given value. The limit has many applications in calculus as well as in other branches of Mathematics.

We have discussed about a function and its graph in first semester. We now give the definition of function, domain and range of a function and value of the function at a point as a review.

Function

Let X and Y be two non-empty sets. A function f from a set X to set Y is a rule which associates each element of X with a unique element of Y . The set X is called the **domain of f** and the set of values of $f(x)$ for every $x \in X$ is called the **range of f** and the set Y is called the **co-domain of f** . In symbol, we write $f: X \rightarrow Y$ in order to mean f is a function from X to Y .

Value of a Function

Let $f: X \rightarrow Y$ be a function and $x = a$ be an element in X . The value of $f(a)$ is called the value of the function f at $x = a$. If $f(a)$ is a finite number then we say that $f(x)$ exists or is defined at $x = a$. If $f(a)$ is not finite, we say $f(x)$ does not exist or is not defined at $x = a$.

Examples:

- The function $f(x) = 2x + 1$ is defined at $x = 2$ since $f(2) = 2 \times 2 + 1 = 5$ which is a finite number.

ii. The function $f(x) = \frac{x^2-4}{x-2}$ is not defined at $x=2$ since

$$f(2) = \frac{2^2-4}{2-2} = \frac{0}{0}, \text{ which is meaningless.}$$

Thus, $f(x)$ is not defined at $x=2$.

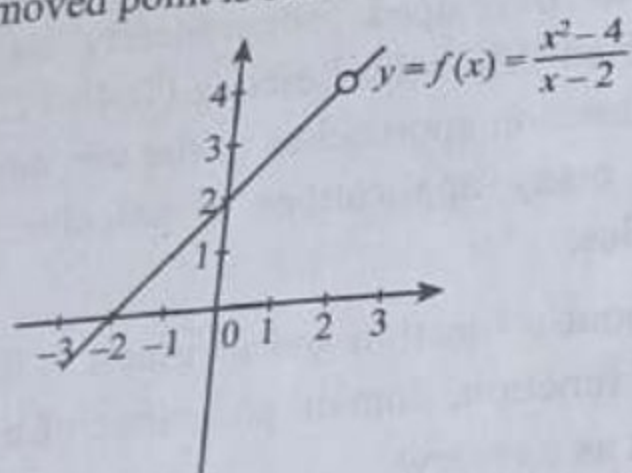
For any $x \neq 2$, we can factorize numerator and cancel the common factor from numerator and denominator.

$$y = f(x) = \frac{x^2-4}{x-2}$$

$$= \frac{(x-2)(x+2)}{x-2}$$

$$= x+2 \text{ for } x \neq 2$$

Thus, the graph of f is the line $y = x+2$ with one point removed; namely the point $(2, 4)$. This removed point is shown as a 'hole' in the figure.



If we give to x , a value not exactly 2 but slightly more than or less than 2 then the value of $f(x)$ is slightly more than or less than 4 respectively.

Value of x more than 2	$f(x) = \frac{x^2-4}{x-2} = x+2, x \neq 2$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001
...	...
Value of x less than 2	$f(x) = \frac{x^2-4}{x-2} = x+2, x \neq 2$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999
...	...

Hence, as the value of x approaches 2, the value of $f(x)$ approaches 4.

Symbolically, we write as $x \rightarrow 2, f(x) \rightarrow 4$.

Meaning of $x \rightarrow a$

Let x be a variable and a be a constant. If x assumes the value nearer and nearer to a i.e. the absolute value of difference between x and a is very small, then we say that ' x tends to a ' or ' x approaches to a '. Symbolically, we write $x \rightarrow a$.

1.2 Indeterminate Forms

If a function $f(x)$ takes the form $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0^\infty, 0^0, 1^\infty$ for some value of x , then these forms are called indeterminate forms.

1.3 Limit of a Function

Definition

Informal Definition of Limit

Let $f(x)$ be defined on an open interval about a , except possibly at a itself. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to a , we say that f approaches the limit L as x approaches a , and we write $\lim_{x \rightarrow a} f(x) = L$.

This definition is informal because the phrases arbitrarily close and sufficiently close are imprecise as their meaning depends on the context.

Working Rules for Finding $\lim_{x \rightarrow a} f(x)$

Rule I (Direct substitution): Put $x = a$ in the given function $f(x)$. If $f(a)$ is finite number then $\lim_{x \rightarrow a} f(x) = f(a)$.

Examples:

i. $\lim_{x \rightarrow 2} (3) = 3$

ii. $\lim_{x \rightarrow -1} (2) = 2$

iii. $\lim_{x \rightarrow 2} (3x^2 + 2) = 3 \times 2^2 + 2 = 14$

If $f(a)$ takes an indeterminate form, we adopt the following rules.

Rule II (Canceling a common factor): If $f(x)$ is a rational function then factorize the numerator and denominator. Cancel out the common factors and put $x = a$.

Example 1. a. $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$
 b. $\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}$

Solution

a. When $x = 2$, $\frac{x^2 - 4}{x - 2} = \frac{2^2 - 4}{2 - 2} = \frac{0}{0}$ which is an indeterminate form.

Now,

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} &= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} (x + 2) \\ &= 2 + 2 \\ &= 4. \end{aligned}$$

b. The function takes the form $\frac{0}{0}$ when $x = 1$.

So,

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 2)}{x(x - 1)} \\ &= \lim_{x \rightarrow 1} \frac{x + 2}{x} \\ &= \frac{1 + 2}{1} \\ &= 3. \end{aligned}$$

Rule III (Creating and canceling a common factor): If the given function contains a surd then simplify it by using conjugate surds. Put $x = a$ after simplification.

Example 2. Evaluate: $\lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$

Solution

The given function takes $\frac{0}{0}$ form when $x = 0$.

Now,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} \times \frac{\sqrt{2+x} + \sqrt{2}}{\sqrt{2+x} + \sqrt{2}} \\ &= \lim_{x \rightarrow 0} \frac{2+x-2}{x(\sqrt{2+x} + \sqrt{2})} \\ &= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{2+x} + \sqrt{2})} \\ &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{2+x} + \sqrt{2}} \\ &= \frac{1}{\sqrt{2} + \sqrt{2}} \\ &= \frac{1}{2\sqrt{2}}. \end{aligned}$$

[We can cancel x as $x \rightarrow 0$ but not $x = 0$.]

Rule IV: Use of other standard formulae and series expansion.

1.4 Algebraic Properties of Limit

Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x \rightarrow a} f(x) = \ell$ and $\lim_{x \rightarrow a} g(x) = m$. Then, we have the following theorems on limits.

1. Sum Rule:

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = \ell + m.$$

2. Difference Rule:

$$\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) = \ell - m.$$

3. Product Rule:

$$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = \ell \cdot m.$$

4. Constant Multiple Rule:

$$\lim_{x \rightarrow a} [kf(x)] = k \lim_{x \rightarrow a} f(x) = k\ell, \text{ where } k \text{ is any constant.}$$

5. Quotient Rule:

$$\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{\ell}{m}, m \neq 0.$$

6. **Power Rule:** If p and q are integers, then

$$\lim_{x \rightarrow a} [f(x)]^q = \left[\lim_{x \rightarrow a} f(x) \right]^q = \ell^q, \text{ provided that } \ell^q \text{ is a real number.}$$

Example 3. Given $\lim_{x \rightarrow 0} f(x) = 1$ and $\lim_{x \rightarrow 0} g(x) = -5$. Calculate

$$\lim_{x \rightarrow 0} \frac{2f(x) - g(x)}{(f(x) + 7)^{\frac{2}{3}}}$$

Solution

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{2f(x) - g(x)}{(f(x) + 7)^{\frac{2}{3}}} &= \frac{\lim_{x \rightarrow 0} \{2f(x) - g(x)\}}{\lim_{x \rightarrow 0} (f(x) + 7)^{\frac{2}{3}}} \\ &= \frac{2 \lim_{x \rightarrow 0} f(x) - \lim_{x \rightarrow 0} g(x)}{\left(\lim_{x \rightarrow 0} f(x) + 7 \right)^{\frac{2}{3}}} \\ &= \frac{(2)(1) - (-5)}{(1 + 7)^{\frac{2}{3}}} \\ &= \frac{2 + 5}{8^{\frac{2}{3}}} = \frac{7}{2^{3 \times \frac{2}{3}}} \\ &= \frac{7}{4} \end{aligned}$$

1.5 Theorem on Limits

For all rational values of n ,

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \text{ provided that 'a' is positive.}$$

Proof

The proof of this theorem has the following three cases:

Case I: When n is a positive integer. By actual division, we have,

$$\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1}$$

$$\begin{aligned} \therefore \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \lim_{x \rightarrow a} [x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1}] \\ &= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \dots + a^{n-1} \\ &= a^{n-1} + a^{n-1} + a^{n-1} + \dots + a^{n-1} \text{ (n terms)} \\ &= na^{n-1} \end{aligned}$$

Case II: When n is a negative integer.

Let $n = -m$ where m is a positive integer. Then,

$$\begin{aligned} \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \lim_{x \rightarrow a} \frac{x^{-m} - a^{-m}}{x - a} \\ &= \lim_{x \rightarrow a} \frac{\frac{1}{x^m} - \frac{1}{a^m}}{x - a} \\ &= \lim_{x \rightarrow a} \frac{a^m - x^m}{x^m a^m (x - a)} \\ &= \lim_{x \rightarrow a} \left[\frac{(x^m - a^m)}{x - a} \cdot \frac{1}{x^m a^m} \right] \\ &= \left(\lim_{x \rightarrow a} \frac{x^m - a^m}{x - a} \right) \left(\lim_{x \rightarrow a} \frac{1}{x^m a^m} \right) \\ &= -ma^{m-1} \frac{1}{a^m a^m} \text{ [Using case (I)]} \\ &= -ma^{m-1-2m} \\ &= (-m)a^{(-m)-1} = na^{n-1} [\because -m = n] \end{aligned}$$

Case III: When n is a rational fraction.

Let $n = \frac{p}{q}$ where p and q are integers and $q \neq 0$.

Now,

$$\begin{aligned} \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \lim_{x \rightarrow a} \frac{x^{p/q} - a^{p/q}}{x - a} \\ &= \lim_{x \rightarrow a} \frac{(x^{1/q})^p - (a^{1/q})^p}{x - a} \end{aligned}$$

Put $x^{1/q} = y$ and $a^{1/q} = b$.

Then $x = y^q$ and $a = b^q$. Also when $x \rightarrow a, y \rightarrow b$

$$\begin{aligned} \therefore \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \lim_{y \rightarrow b} \frac{y^p - b^p}{y^q - b^q} \\ &= \lim_{y \rightarrow b} \frac{\left(\frac{y^p - b^p}{y - b} \right)}{\left(\frac{y^q - b^q}{y - b} \right)} = \frac{\lim_{y \rightarrow b} \frac{y^p - b^p}{y - b}}{\lim_{y \rightarrow b} \frac{y^q - b^q}{y - b}} \end{aligned}$$

$$= \frac{pb^{p-1}}{qb^{q-1}}$$

[Using case I or II depending upon the sign of p and

$$= \frac{p}{q} b^{p/q-1}$$

$$= \frac{p}{q} (b^q)^{p/q-1}$$

$$= n a^{n-1}$$

Hence for all rational values of n , $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1}$.

The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing a except possibly at $x = a$ itself. Suppose also that

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = \ell$$

Then, $\lim_{x \rightarrow a} f(x) = \ell$.

Example 4. Given that: $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$ for all $x \neq 0$, find $\lim_{x \rightarrow 0} u(x)$.

Solution

$$\lim_{x \rightarrow 0} 1 - \frac{x^2}{4} = 1 - \frac{0}{4} = 1$$

$$\text{and } \lim_{x \rightarrow 0} 1 + \frac{x^2}{2} = 1 + \frac{0}{2} = 1$$

By Sandwich theorem, we have,

$$\lim_{x \rightarrow 0} u(x) = 1.$$

1.6 Limits at Infinity ($x \rightarrow \infty$ or $-\infty$)

A function $f(x)$ is said to tend to a number L (called the limit) as x tends to ∞ (or $-\infty$) if the numerical difference between $f(x)$ and L can be made as small as we please by taking x large (or small) enough, i.e., for all $\varepsilon > 0$, there exists K such that $|f(x) - L| < \varepsilon$ whenever $x > K$ (or $x < -K$). It is denoted by $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$.

Example 5. Evaluate:

a. $\lim_{x \rightarrow \infty} \frac{1}{x}$

b. $\lim_{x \rightarrow -\infty} \frac{1}{x^2}$

c. $\lim_{x \rightarrow \infty} \frac{3x-5}{4x+1}$

Solution

a. $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$.

b. $\lim_{x \rightarrow -\infty} \frac{1}{x^2} = 0$.

c. $\lim_{x \rightarrow \infty} \frac{3x-5}{4x+1} = \lim_{x \rightarrow \infty} \frac{3 - \frac{5}{x}}{4 + \frac{1}{x}}$

$$= \frac{\lim_{x \rightarrow \infty} (3 - \frac{5}{x})}{\lim_{x \rightarrow \infty} (4 + \frac{1}{x})}$$

$$= \frac{3-0}{4+0}$$

$$= \frac{3}{4}.$$

1.7 Infinite Limits

Let $f(x)$ be a function. If the absolute value of $f(x)$ ultimately becomes greater than any assignable magnitude as $x \rightarrow a$, however large then we say that $f(x)$ has infinite limits. In such case, we write

$$\lim_{x \rightarrow a} f(x) = \infty \quad \text{or,} \quad \lim_{x \rightarrow a} f(x) = -\infty.$$

Note that the symbol " ∞ " generally read "infinity" is not a number, but is a symbol indicating a mode of limiting behaviour.

Illustrative Examples

Example 1. Evaluate: $\lim_{x \rightarrow 1} \frac{3x+5}{2x-1}$.

Solution

When x approaches 1, $3x + 5$ approaches $3 \times 1 + 5 = 8$ and $2x - 1$ approaches $2 \times 1 - 1 = 1$.

Hence, $\lim_{x \rightarrow 1} \frac{3x+5}{2x-1} = \frac{8}{1} = 8.$

Example 2. Evaluate: $\lim_{x \rightarrow 0} \frac{7x^2+5x}{3x}.$

Solution

When $x = 0$, the function $\frac{7x^2+5x}{3x}$ takes the form $\frac{0}{0}$ which is indeterminate form. But, we are not finding the value of the function at $x = 0$. Here, we have to find the number that the given function approaches as x approaches to 0. Note that $x \rightarrow 0$ does not mean $x = 0$. Since $x \neq 0$, we can cancel x from the numerator and the denominator. Therefore

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{7x^2+5x}{3x} &= \lim_{x \rightarrow 0} \frac{x(7x+5)}{3x} \\ &= \lim_{x \rightarrow 0} \frac{(7x+5)}{3} \\ &= \frac{7 \times 0 + 5}{3} \\ &= \frac{5}{3}.\end{aligned}$$

Example 3. Evaluate: $\lim_{x \rightarrow 1} \frac{x^2-5x+4}{x^2+6x-7}.$

Solution

When $x = 1$, the given function takes the indeterminate form $\frac{0}{0}$.

Now,

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^2-5x+4}{x^2+6x-7} &= \lim_{x \rightarrow 1} \frac{x^2-4x-x+4}{x^2+7x-x-7} \\ &= \lim_{x \rightarrow 1} \frac{x(x-4)-1(x-4)}{x(x+7)-1(x+7)} \\ &= \lim_{x \rightarrow 1} \frac{(x-4)(x-1)}{(x-1)(x+7)} \\ &= \lim_{x \rightarrow 1} \frac{(x-4)}{(x+7)} \\ &= \frac{1-4}{1+7} \\ &= -\frac{3}{8}.\end{aligned}$$

Example 4. Evaluate: $\lim_{x \rightarrow 2} \frac{x^4-16}{x-2}.$

Solution

When $x = 2$, the function $\frac{x^4-16}{x-2}$ takes the form $\frac{0}{0}$ which is an indeterminate form.

So,

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^4-16}{x-2} &= \lim_{x \rightarrow 2} \frac{x^4-2^4}{x-2} \\ &= 4(2)^{4-1} \quad \left[\because \lim_{x \rightarrow a} \frac{x^n-a^n}{x-a} = n a^{n-1} \right] \\ &= 4(2)^3 \\ &= 32.\end{aligned}$$

Example 5. Evaluate: $\lim_{z \rightarrow 1} \frac{z^{15}-1}{z^{10}-1}.$

Solution

When $z = 1$, the function $\frac{z^{15}-1}{z^{10}-1}$ takes the form $\frac{0}{0}$ which is an indeterminate form.

So,

$$\begin{aligned}\lim_{z \rightarrow 1} \frac{z^{15}-1}{z^{10}-1} &= \lim_{z \rightarrow 1} \frac{\left(\frac{z^{15}-1}{z-1}\right)}{\left(\frac{z^{10}-1}{z-1}\right)} \\ &= \frac{\lim_{z \rightarrow 1} \frac{z^{15}-1}{z-1}}{\lim_{z \rightarrow 1} \frac{z^{10}-1}{z-1}} \\ &= \frac{15(1)^{14}}{10(1)^9} \quad \left[\because \lim_{x \rightarrow a} \frac{x^n-a^n}{x-a} = n a^{n-1} \right] \\ &= \frac{15}{10} \\ &= \frac{3}{2}.\end{aligned}$$

Example 6. Evaluate: $\lim_{x \rightarrow a} \frac{(x+3)^{3/2} - (a+3)^{3/2}}{x-a}.$

Solution

When $x = a$, the given function takes the indeterminate form $\frac{0}{0}$.

$$\begin{aligned} \text{So, } \lim_{x \rightarrow a} \frac{(x+3)^{3/2} - (a+3)^{3/2}}{x-a} &= \lim_{(x+3) \rightarrow (a+3)} \frac{(x+3)^{3/2} - (a+3)^{3/2}}{(x+3) - (a+3)} \\ &= \frac{3}{2} (a+3)^{(3/2-1)} \left[\because \lim_{x \rightarrow a} \frac{x^n - a^n}{x-a} = n a^{n-1} \right] \\ &= \frac{3}{2} (a+3)^{1/2} \end{aligned}$$

Example 7. Evaluate: $\lim_{x \rightarrow 0} \frac{\sqrt{1+2x} - \sqrt{1-2x}}{x}$

Solution

When $x = 0$, the given function takes the indeterminate form $\frac{0}{0}$.

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow 0} \frac{\sqrt{1+2x} - \sqrt{1-2x}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{1+2x} - \sqrt{1-2x}}{x} \times \frac{\sqrt{1+2x} + \sqrt{1-2x}}{\sqrt{1+2x} + \sqrt{1-2x}} \\ &= \lim_{x \rightarrow 0} \frac{1+2x - 1+2x}{x(\sqrt{1+2x} + \sqrt{1-2x})} \\ &= \lim_{x \rightarrow 0} \frac{4x}{x(\sqrt{1+2x} + \sqrt{1-2x})} \\ &= \lim_{x \rightarrow 0} \frac{4}{\sqrt{1+2x} + \sqrt{1-2x}} \\ &= \frac{4}{1+1} = 2 \end{aligned}$$

Example 8. Prove that $\lim_{x \rightarrow 1} \frac{x - \sqrt{2-x^2}}{2x - \sqrt{2+2x^2}} = 2$.

Solution

When $x = 1$, the given function takes the indeterminate form $\frac{0}{0}$.

$$\begin{aligned} \text{L.H.S.} &= \lim_{x \rightarrow 1} \frac{x - \sqrt{2-x^2}}{2x - \sqrt{2+2x^2}} \\ &= \lim_{x \rightarrow 1} \frac{x - \sqrt{2-x^2}}{2x - \sqrt{2+2x^2}} \times \frac{2x + \sqrt{2+2x^2}}{2x + \sqrt{2+2x^2}} \times \frac{x + \sqrt{2-x^2}}{x + \sqrt{2-x^2}} \\ &= \lim_{x \rightarrow 1} \frac{(x^2 - 2 + x^2)(2x + \sqrt{2+2x^2})}{(4x^2 - 2 - 2x^2)(x + \sqrt{2-x^2})} \\ &= \lim_{x \rightarrow 1} \frac{(2x^2 - 2)(2x + \sqrt{2+2x^2})}{(2x^2 - 2)(x + \sqrt{2-x^2})} \end{aligned}$$

$$\begin{aligned} &= \lim_{x \rightarrow 1} \frac{2x + \sqrt{2+2x^2}}{x + \sqrt{2-x^2}} \\ &= \frac{2 \times 1 + \sqrt{2+2 \times 1^2}}{1 + \sqrt{2-1^2}} \\ &= \frac{2+2}{1+1} \\ &= 2 \\ &= \text{R.H.S.} \end{aligned}$$

Example 9. Evaluate: $\lim_{x \rightarrow \infty} \frac{x^2 - 5x + 7}{x^2 + 2x + 3}$

Solution

$$\lim_{x \rightarrow \infty} \frac{x^2 - 5x + 7}{x^2 + 2x + 3} \quad \left[\frac{\infty}{\infty} \text{ form} \right]$$

Dividing both numerator and denominator by x^2 .

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \frac{1 - \frac{5}{x} + \frac{7}{x^2}}{1 + \frac{2}{x} + \frac{3}{x^2}} \\ &= \frac{1-0+0}{1+0+0} = 1. \end{aligned}$$

Example 10. Evaluate $\lim_{x \rightarrow \infty} \sqrt{x}(\sqrt{x} - \sqrt{x-a})$.

Solution

$$\begin{aligned} &\lim_{x \rightarrow \infty} \sqrt{x}(\sqrt{x} - \sqrt{x-a}) \quad [\infty - \infty \text{ form}] \\ &= \lim_{x \rightarrow \infty} \sqrt{x} \frac{(\sqrt{x} - \sqrt{x-a})}{1} \times \frac{\sqrt{x} + \sqrt{x-a}}{\sqrt{x} + \sqrt{x-a}} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{x}(x - x + a)}{(\sqrt{x} + \sqrt{x-a})} \\ &= \lim_{x \rightarrow \infty} \frac{a\sqrt{x}}{\sqrt{x} + \sqrt{x-a}} \end{aligned}$$

Dividing both numerator and denominator by $\sqrt{x}$

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \left(\frac{\frac{a\sqrt{x}}{\sqrt{x}}}{\frac{\sqrt{x}}{\sqrt{x}} + \frac{\sqrt{x-a}}{\sqrt{x}}} \right) \\ &= \lim_{x \rightarrow \infty} \left(\frac{a}{1 + \sqrt{\frac{x-a}{x}}} \right) \end{aligned}$$

$$\begin{aligned}
 &= \lim_{x \rightarrow \infty} \left(\frac{a}{1 + \sqrt{\frac{x-a}{x-x}}} \right) \\
 &= \lim_{x \rightarrow \infty} \left(\frac{a}{1 + \sqrt{1 - \frac{a}{x}}} \right) \\
 &= \frac{a}{1 + \sqrt{1-0}} \\
 &= \frac{a}{2}.
 \end{aligned}$$

Example 11. Evaluate: $\lim_{x \rightarrow \infty} \frac{2\sqrt{1+x} - \sqrt{2x-1}}{x}$.

Solution

$$\begin{aligned}
 &\text{Here, } \lim_{x \rightarrow \infty} \frac{2\sqrt{1+x} - \sqrt{2x-1}}{x} \\
 &= \lim_{x \rightarrow \infty} \frac{2\sqrt{1+x} - \sqrt{2x-1}}{x} \times \frac{2\sqrt{1+x} + \sqrt{2x-1}}{2\sqrt{1+x} + \sqrt{2x-1}} \\
 &= \lim_{x \rightarrow \infty} \frac{(2\sqrt{1+x})^2 - (\sqrt{2x-1})^2}{x(2\sqrt{1+x} + \sqrt{2x-1})} \\
 &= \lim_{x \rightarrow \infty} \frac{4(1+x) - (2x-1)}{x(2\sqrt{1+x} + \sqrt{2x-1})} \\
 &= \lim_{x \rightarrow \infty} \frac{4 + 4x - 2x + 1}{x(2\sqrt{1+x} + \sqrt{2x-1})} \\
 &= \lim_{x \rightarrow \infty} \frac{2x + 5}{x(2\sqrt{1+x} + \sqrt{2x-1})}
 \end{aligned}$$

Dividing both numerator and denominator by x .

$$\begin{aligned}
 &= \lim_{x \rightarrow \infty} \frac{2 + \frac{5}{x}}{2\sqrt{1+x} + \sqrt{2x-1}} \\
 &= \frac{2}{\infty} = 0.
 \end{aligned}$$

Exercise 1.1

1. Evaluate the following limits.

a. $\lim_{x \rightarrow 1} (2x+3)$

c. $\lim_{x \rightarrow 5} \frac{4}{x-7}$

b. $\lim_{x \rightarrow -7} (2x+5)$

d. $\lim_{h \rightarrow 0} \frac{5}{\sqrt{2h+1}+1}$

2. Evaluate

a. $\lim_{x \rightarrow 0} \frac{7x^2+4x}{x}$

c. $\lim_{x \rightarrow 1} \frac{x^2-4x+3}{x-1}$

e. $\lim_{x \rightarrow 1} \frac{x^2+x-2}{x^2-1}$

g. $\lim_{x \rightarrow -3} \frac{x+3}{x^2+4x+3}$

b. $\lim_{x \rightarrow 5} \frac{x^2-25}{x-5}$

d. $\lim_{x \rightarrow 2} \frac{x^2-7x+10}{x-2}$

f. $\lim_{x \rightarrow 1} \frac{x^2-4x+3}{x^2-3x+2}$

h. $\lim_{x \rightarrow 3} \left[\frac{1}{x-3} - \frac{9}{x^2-3x^2} \right]$

3. Evaluate

a. $\lim_{x \rightarrow 9} \frac{\sqrt{x}-3}{x-9}$

c. $\lim_{x \rightarrow 0} \frac{7x}{\sqrt{3x+4}-2}$

e. $\lim_{x \rightarrow 4} \frac{x^2-16}{\sqrt{3x+4}-4}$

g. $\lim_{x \rightarrow a} \frac{\sqrt{3a-x}-\sqrt{x+a}}{4(x-a)}$

i. $\lim_{x \rightarrow 2} \frac{x-\sqrt{8-x^2}}{\sqrt{x^2+12}-4}$

b. $\lim_{x \rightarrow 0} \frac{\sqrt{x+4}-2}{x}$

d. $\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x+3}-2}$

f. $\lim_{x \rightarrow -1} \frac{\sqrt{x^2+8}-3}{x+1}$

h. $\lim_{x \rightarrow a} \frac{\sqrt{2x}-\sqrt{3x-a}}{\sqrt{x}-\sqrt{a}}$

4. Evaluate

a. $\lim_{x \rightarrow a} \frac{x^5-a^5}{x-a}$

c. $\lim_{x \rightarrow a} \frac{x^{\frac{2}{3}}-a^{\frac{2}{3}}}{x-a}$

b. $\lim_{x \rightarrow 2} \frac{x^8-256}{x-2}$

d. $\lim_{x \rightarrow a} \frac{x^{\frac{5}{3}}-a^{\frac{5}{3}}}{x^{\frac{2}{3}}-a^{\frac{2}{3}}}, a > 0$

5. If $\lim_{x \rightarrow 1} \frac{x^4-1}{x-1} = \lim_{x \rightarrow k} \frac{x^3-k^3}{x^2-k^2}$, find the value of k .

6. Compute

a. $\lim_{x \rightarrow \infty} \frac{x^2+7x+3}{9x^2+7x+2}$

c. $\lim_{x \rightarrow \infty} \frac{x^2+x-2}{4x^3-1}$

b. $\lim_{x \rightarrow \infty} \frac{4x^2+x+1}{3x^2+2x+1}$

d. $\lim_{x \rightarrow \infty} (\sqrt{x+1}-\sqrt{x})$

$$e. \lim_{x \rightarrow \infty} \sqrt{x}(\sqrt{x+1} - \sqrt{x}) \quad f. \lim_{x \rightarrow \infty} \frac{\sqrt{x^2+1}}{x+1}$$

7. If $f(x) = \frac{ax+b}{x-5}$, $\lim_{x \rightarrow 0} f(x) = -1$ and $\lim_{x \rightarrow \infty} f(x) = 3$, find the value of $f(3)$.

Very Short Answer Questions

- What do you mean by $x \rightarrow a$?
- Write any two indeterminate forms.
- What is limit?
- If $\lim_{x \rightarrow a} f(x) = -2$ and $\lim_{x \rightarrow a} g(x) = 3$, then find $\lim_{x \rightarrow a} f(x)g(x)$.
- What is the value of $\lim_{x \rightarrow 2} (4)$?
- Evaluate $\lim_{t \rightarrow 1} \left(\frac{4}{t-9} \right)$.
- Write the formula of $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a}$, where n is a rational number.
- Evaluate $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$.
- What is the value of $\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x}$?
- Evaluate $\lim_{x \rightarrow \infty} \frac{3x^2 + 4x - 1}{4x^2 + 5}$.
- If $\lim_{x \rightarrow 2} \frac{x^n - 2^n}{x - 2} = 80$ then find the values of n .
- When a function $f(x)$ is said to be infinitesimal at $x = a$?

Answers

- a. 5 b. -9 c. -2 d. $\frac{5}{2}$
- a. 4 b. 10 c. -2 d. -3
- e. $\frac{3}{2}$ f. 2 g. $-\frac{1}{2}$ h. $\frac{2}{3}$
- a. $\frac{1}{6}$ b. $\frac{1}{4}$ c. $\frac{28}{3}$ d. 4
- e. $\frac{64}{3}$ f. $-\frac{1}{3}$ g. $-\frac{1}{4\sqrt{2}a}$ h. $-\frac{1}{\sqrt{2}}$
- i. 4

$$4. a. 5a^4 \quad b. 1024$$

$$5. \frac{8}{3}$$

$$6. a. \frac{1}{9} \quad b. \frac{4}{3}$$

$$e. \frac{1}{2} \quad f. 1$$

$$7. -7$$

Very Short Answer Questions

$$4. -6$$

$$5. 4$$

$$9. \frac{1}{2}$$

$$10. \frac{3}{4}$$

$$6. -\frac{1}{2}$$

$$8. 6$$

$$11. 5$$

1.8 Limits of Trigonometric Functions

Review of Some Important Formulae

Trigonometric ratios of an acute angle θ in right angled triangle shown in the figure are as follows:

$$\sin \theta = \frac{p}{h}$$

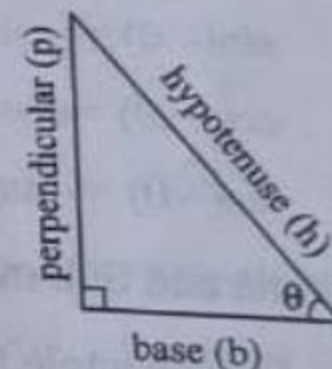
$$\operatorname{cosec} \theta = \frac{h}{p}$$

$$\cos \theta = \frac{b}{h}$$

$$\sec \theta = \frac{h}{b}$$

$$\tan \theta = \frac{p}{b}$$

$$\cot \theta = \frac{b}{p}$$



Elementary Identities

- $\sin \theta \cdot \operatorname{cosec} \theta = 1$
- $\cos \theta \cdot \sec \theta = 1$
- $\tan \theta \cdot \cot \theta = 1$

Rule of Signs of Trigonometric Function

Quadrant	Function		
	sin	cos	tan
I	+	+	+
II	+	-	-
III	-	-	+
IV	-	+	-

Pythagorean Identities

- $\sin^2 \theta + \cos^2 \theta = 1$
- $\sec^2 \theta - \tan^2 \theta = 1$
- $\operatorname{cosec}^2 \theta - \cot^2 \theta = 1$

Addition and Subtraction Formulae

- $\sin(A+B) = \sin A \cos B + \cos A \sin B$
- $\sin(A-B) = \sin A \cos B - \cos A \sin B$
- $\cos(A+B) = \cos A \cos B - \sin A \sin B$
- $\cos(A-B) = \cos A \cos B + \sin A \sin B$
- $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
- $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Also,

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = -\tan \theta$$

Multiple and Sub-multiple Angle Formulae

a. Double angle formula

- $\sin 2A = 2 \sin A \cos A$
- $\cos 2A = \cos^2 A - \sin^2 A$
 $= 2 \cos^2 A - 1$
 $= 1 - 2 \sin^2 A$
- $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

$$\text{Also, } 1 - \cos 2A = 2 \sin^2 A$$

$$1 + \cos 2A = 2 \cos^2 A$$

b. Half angle formulae

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$$

$$\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} = 2 \cos^2 \frac{A}{2} - 1 = 1 - 2 \sin^2 \frac{A}{2}$$

$$\tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$$

$$\text{Also, } 1 - \cos A = 2 \sin^2 \frac{A}{2}$$

$$1 + \cos A = 2 \cos^2 \frac{A}{2}$$

c. Triple angle formulae

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

Transformation Formulae

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$$

$$\sin(A+B) - \sin(A-B) = 2 \cos A \sin B$$

$$\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$$

$$\cos(A-B) - \cos(A+B) = 2 \sin A \sin B$$

Also,

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$$

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$$

$$\begin{aligned} \cos C - \cos D &= 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2} \\ &= -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2} \end{aligned}$$

Theorem 1:

$$\text{i. } \lim_{\theta \rightarrow 0} \sin \theta = 0$$

$$\text{ii. } \lim_{\theta \rightarrow 0} \cos \theta = 1.$$

Proof

Consider a right angled triangle ABC in which $\angle B$ is a right angle and $\angle ACB = \theta$ radians. Then

$$\sin \theta = \frac{BA}{CA} \text{ and } \cos \theta = \frac{CB}{CA}$$

Let us keep BC fixed and go on decreasing θ , we find that A comes nearer and nearer to C.

$\therefore$ when $\theta \rightarrow 0$, $A \rightarrow B$

$\therefore \theta \rightarrow 0 \Rightarrow BA \rightarrow 0$ and $CA \rightarrow CB$.

$$\Rightarrow \frac{BA}{CA} \rightarrow 0 \text{ and } \frac{CB}{CA} \rightarrow 1$$

$$\Rightarrow \sin \theta \rightarrow 0 \text{ and } \cos \theta \rightarrow 1$$

Hence, we have

$$\text{i. } \lim_{\theta \rightarrow 0} \sin \theta = 0$$

$$\text{ii. } \lim_{\theta \rightarrow 0} \cos \theta = 1$$

As in the case of algebraic functions, trigonometric functions also takes indeterminate form $\frac{0}{0}$ at some points but may have limit at those points. To determine the limiting values of such trigonometric functions, the limiting value of $\frac{\sin \theta}{\theta}$ as $\theta \rightarrow 0$ plays a very helpful role.

So, we find geometrically the limiting value of $\frac{\sin \theta}{\theta}$ as $\theta \rightarrow 0$.

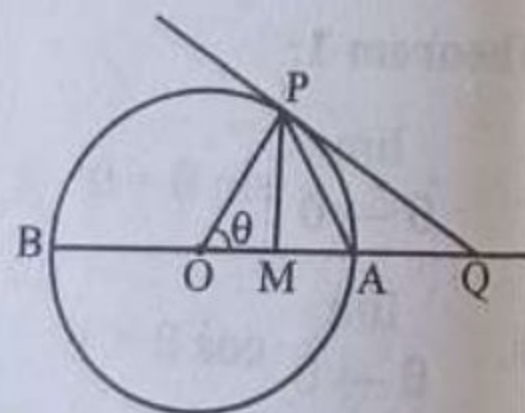
Theorem 2:

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \text{ where } \theta \text{ is in radian.}$$

Proof

Let ABP be a circle of radius r and centre at O. Let AP be an arc which subtends an angle θ at the centre O of the circle. Let PQ be a tangent at P which meets BA produced at Q. Join PA and draw $PM \perp BA$. Then,

$$\text{Area of } \triangle OPA \leq \text{Area of Sector OPA} \leq \text{Area of } \triangle OPQ \dots (i)$$



Now,

$$\text{Area of } \triangle OPA = \frac{1}{2} OA \cdot PM$$

$$= \frac{1}{2} r \cdot r \sin \theta \quad \left[\begin{array}{l} \because OA = r \\ \text{and in right angled } \triangle POM, \frac{PM}{OP} = \sin \theta \\ \Rightarrow PM = OP \sin \theta = r \sin \theta \end{array} \right]$$

$$= \frac{1}{2} r^2 \sin \theta$$

$$\text{Area of Sector OPA} = \left(\frac{\theta}{2\pi} \right) \pi r^2 = \frac{1}{2} r^2 \theta$$

$$\text{Area of } \triangle OPQ = \frac{1}{2} OP \cdot PQ$$

$$= \frac{1}{2} r \cdot r \tan \theta \quad \left[\begin{array}{l} OP = r \\ \text{and in right angled } \triangle OPQ, \tan \theta = \frac{PQ}{OP} \\ \Rightarrow PQ = r \tan \theta \end{array} \right]$$

Substituting these values in (i), we get,

$$\frac{1}{2} r^2 \sin \theta \leq \frac{1}{2} r^2 \theta \leq \frac{1}{2} r^2 \tan \theta$$

$$\text{or, } \sin \theta \leq \theta \leq \tan \theta$$

$$\text{or, } 1 \leq \frac{\theta}{\sin \theta} \leq \frac{1}{\cos \theta} \quad (\text{Dividing throughout by } \sin \theta)$$

$$\text{or, } 1 \geq \frac{\sin \theta}{\theta} \geq \cos \theta$$

Taking limit as $\theta \rightarrow 0$ on all sides,

$$\text{or, } \lim_{\theta \rightarrow 0} 1 \geq \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \geq \lim_{\theta \rightarrow 0} \cos \theta$$

$$\text{or, } 1 \geq \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \geq 1$$

$$\therefore \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$

Alternatively (Using Sine Series)

$$\text{Using Sine Series also we can prove } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

We have, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

NOTE $n!$ read as n factorial means multiplication of all natural numbers from n to 1.

Now,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin x}{x} &= \lim_{x \rightarrow 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)}{x} \\ &= \lim_{x \rightarrow 0} \frac{x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right)}{x} \\ &= 1.\end{aligned}$$

1.9 Limits of Exponential and Logarithmic Functions

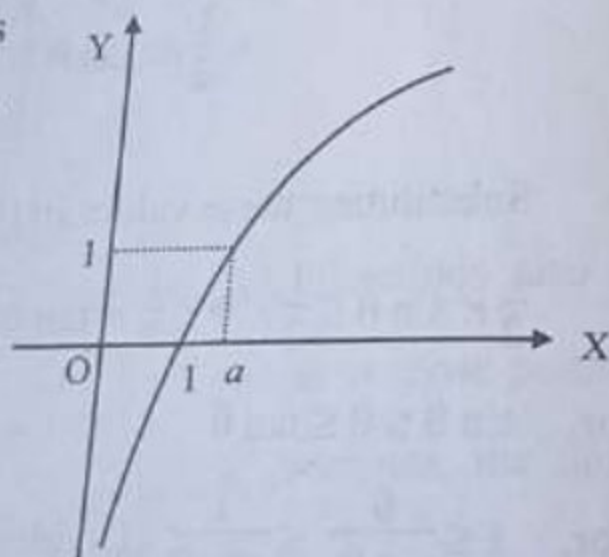
The exponential and logarithmic functions are defined in *First semester*. If $y = a^x$,

$a > 1$ then $\lim_{x \rightarrow -\infty} a^x = 0$ and

$\lim_{x \rightarrow \infty} a^x = \infty$ for $a < 1$. If $y = \log_a x$ is the

logarithmic function $a > 0$ and $a \neq 1$, then

$\lim_{x \rightarrow 0} \log_a x = -\infty$ and $\lim_{x \rightarrow \infty} \log_a x = \infty$, for $a > 1$.



Here, $\log_e a = 1$.

The function $f(x) = e^x$ is called the **natural exponential function**. The number e is defined as

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{h \rightarrow 0} (1 + h)^{1/h},$$

where, $e \approx 2.71828$. Logarithms to the base e are called **natural logarithms**, or **Naperian logarithms**, after John Napier, the inventor of logarithms and we use $\ln x$ instead of $\log_e x$.

We state and prove the following limit theorems on exponential and logarithmic functions.

a. $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$

b. $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$

c. $\lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x}\right) = \log_e a, a > 0.$

Solution

a. At $x = 0$, the given function takes the indeterminate form $\frac{0}{0}$

So,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} &= \lim_{x \rightarrow 0} \ln(1+x)^{1/x} \\ &= \ln \left(\lim_{x \rightarrow 0} (1+x)^{1/x} \right) \\ &= \ln e = 1.\end{aligned}$$

b. Put $e^x - 1 = t$. Then, $x = \ln(1+t)$. Use the result obtained in (a) to get the limiting value.

c. Let $a^x - 1 = t$, then $t \rightarrow 0$ as $x \rightarrow 0$ and $x = \frac{\log_e(t+1)}{\log_e a}$

$$[\because \log_e(1+t) = \log_a(1+t) \cdot \log_e a]$$

$$\begin{aligned}\therefore \lim_{x \rightarrow 0} \frac{a^x - 1}{x} &= \lim_{t \rightarrow 0} \frac{t \log_e a}{\log_e(t+1)} \\ &= \lim_{t \rightarrow 0} \frac{\log_e a}{\log_e(t+1)^{1/t}} \\ &= \frac{\log_e a}{\log_e e} \\ &= \log_e a.\end{aligned}$$

When $a = e$, we get, $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \log_e e = 1.$

Alternative methods

a. $\lim_{x \rightarrow 0} \frac{\log_e(1+x)}{x} = 1$

Since,

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots (-1 < x \leq 1)$$

$$\begin{aligned}\therefore \frac{\log_e(1+x)}{x} &= \frac{1}{x} \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \right) \\ &= 1 - \frac{x}{2} + \frac{x^2}{3} - \dots\end{aligned}$$

So, $\lim_{x \rightarrow 0} \frac{\log_e(1+x)}{x} = \lim_{x \rightarrow 0} \left(1 - \frac{x}{2} + \frac{x^2}{3} - \dots \right) = 1.$

$$b. \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

We have,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\therefore \frac{e^x - 1}{x} = \frac{1}{x} \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots - 1 \right)$$

$$= \frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \dots$$

Now,

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{x \rightarrow 0} \left(\frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \dots \right) = 1$$

$$\text{Hence, } \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1.$$

$$c. \lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x} \right) = \log_e a$$

We have,

$$a^x = 1 + x(\log_e a) + \frac{x^2}{2!} (\log_e a)^2 + \dots$$

$$\therefore \lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x} \right) = \lim_{x \rightarrow 0} \left[\frac{\{1 + x(\log_e a) + \frac{x^2}{2!} (\log_e a)^2 + \dots\} - 1}{x} \right]$$

$$= \lim_{x \rightarrow 0} \frac{x\{(\log_e a) + \frac{x}{2!} (\log_e a)^2 + \dots\}}{x}$$

$$= \lim_{x \rightarrow 0} \{(\log_e a) + \frac{x}{2!} (\log_e a)^2 + \dots\}$$

$$= \log_e a.$$

NOTE We write $\log_e x = \ln x$. Sometimes we may write $\log x$ for $\log_e x$ if there is no confusion.

IMPORTANT FORMULAE OF LOGARITHMIC FUNCTIONS

Let $a > 0$; m, n be any positive integers.

$$a. \log_a mn = \log_a m + \log_a n$$

$$b. \log_a \frac{m}{n} = \log_a m - \log_a n$$

$$c. \log_a m^n = n \log_a m$$

$$d. \log_a m = \log_a b \cdot \log_b m$$

$$e. \log_a a = 1$$

$$\text{Example 1. a. } \lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x}$$

$$b. \lim_{x \rightarrow 0} \left(\frac{5^x - 3^x}{x} \right)$$

$$c. \lim_{x \rightarrow 0} \frac{\log_e (1+x)}{3x}$$

Solution

$$a. \lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \left\{ \left(\frac{e^{3x} - 1}{3x} \right) \times 3 \right\}$$

$$= 3 \times \lim_{3x \rightarrow 0} \frac{e^{3x} - 1}{3x} \quad (\text{As } x \rightarrow 0, 3x \rightarrow 0)$$

$$= 3 \lim_{y \rightarrow 0} \frac{e^y - 1}{y} \quad \text{where } y = 3x$$

$$= 3 \times 1$$

$$= 3. \quad \left[\because \lim_{y \rightarrow 0} \frac{e^y - 1}{y} = 1 \right]$$

$$b. \lim_{x \rightarrow 0} \left(\frac{5^x - 3^x}{x} \right) \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \left\{ \frac{(5^x - 1) - (3^x - 1)}{x} \right\}$$

$$= \lim_{x \rightarrow 0} \left(\frac{5^x - 1}{x} \right) - \lim_{x \rightarrow 0} \left(\frac{3^x - 1}{x} \right)$$

$$= \log_e 5 - \log_e 3 \quad \left[\because \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a \right]$$

$$= \log_e \frac{5}{3}. \quad \left[\because \log_e a - \log_e b = \log_e \frac{a}{b} \right]$$

$$c. \lim_{x \rightarrow 0} \frac{\log_e (1+x)}{3x} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{\log_e (1+x)}{x} \cdot \frac{1}{3}$$

$$= \frac{1}{3} \lim_{x \rightarrow 0} \frac{\log_e (1+x)}{x}$$

$$= \frac{1}{3} \times 1$$

$$= \frac{1}{3}.$$

Illustrative Examples

Example 1. Prove that $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1$.

Solution

When $\theta = 0$, $\frac{\tan \theta}{\theta}$ takes the indeterminate form $\frac{0}{0}$.

So,

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} &= \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{1}{\cos \theta} \right) \\ &= \left(\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \right) \cdot \left(\lim_{\theta \rightarrow 0} \frac{1}{\cos \theta} \right) \\ &= 1 \cdot \frac{1}{\cos 0} \\ &= 1.\end{aligned}$$

Example 2. Evaluate $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$.

Solution

When $x = 0$, $\frac{\sin 3x}{x}$ takes the indeterminate form $\frac{0}{0}$.

So,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 3x}{x} &= \left(\lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times 3 \right) \\ &= 3 \lim_{3x \rightarrow 0} \frac{\sin 3x}{3x} \left[\begin{array}{l} \text{When } \rightarrow 0 \\ \text{then } 3x \rightarrow 0 \end{array} \right] \\ &= 3 \times 1 \\ &= 3.\end{aligned}$$

Example 3. Evaluate $\lim_{x \rightarrow 0} \frac{\sin 7x}{\tan 3x}$.

Solution

When $x = 0$, $\frac{\sin 7x}{\tan 3x}$ takes the indeterminate form $\frac{0}{0}$.

So,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 7x}{\tan 3x} &= \lim_{x \rightarrow 0} \frac{\frac{\sin 7x}{7x} \times 7x}{\frac{\tan 3x}{3x} \times 3x} = \frac{7}{3} \frac{\lim_{7x \rightarrow 0} \frac{\sin 7x}{7x}}{\lim_{3x \rightarrow 0} \frac{\tan 3x}{3x}} \\ &= \frac{7}{3} \times \frac{1}{1} \\ &= \frac{7}{3}.\end{aligned}$$

Example 4. Show that $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$.

Solution

$\frac{1 - \cos x}{x^2}$ takes the indeterminate form $\frac{0}{0}$ when $x = 0$.

Now,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2 \times 4} \\ &= \frac{1}{2} \lim_{x \rightarrow 0} \left\{ \frac{\sin \left(\frac{x}{2}\right)}{\frac{x}{2}} \right\}^2 \\ &= \frac{1}{2} \times 1^2 = \frac{1}{2}.\end{aligned}$$

Example 5. Evaluate $\lim_{x \rightarrow \frac{\pi}{2}} (\sec x - \tan x)$.

Solution

Here, $(\sec x - \tan x)$ takes the indeterminate form $\infty - \infty$ when $x = \frac{\pi}{2}$.

So,

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{2}} (\sec x - \tan x) &= \lim_{x \rightarrow \frac{\pi}{2}} \left[\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right] = \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{1 - \sin x}{\cos x} \right) \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\cos x} \times \frac{1 + \sin x}{1 + \sin x} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin^2 x}{\cos x(1 + \sin x)} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos^2 x}{\cos x(1 + \sin x)} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{1 + \sin x} = \frac{\cos \frac{\pi}{2}}{1 + \sin \frac{\pi}{2}} \\ &= \frac{0}{1 + 1} \\ &= 0.\end{aligned}$$

Example 6. Evaluate: $\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1}$.

Solution

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1} \quad \left[\frac{0}{0} \text{ form} \right] \\ &= \lim_{x \rightarrow 0} \frac{-(1 - \cos 2x)}{-(1 - \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{2 \sin^2 \left(\frac{x}{2}\right)} = \lim_{x \rightarrow 0} \left\{ \frac{\left(\frac{\sin x}{x}\right)^2}{\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2} \cdot \frac{1}{4} \right\} \\ &= 4 \cdot \frac{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x}\right)^2}{\lim_{\frac{x}{2} \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2} \\ &= 4 \cdot \frac{1^2}{1^2} = 4. \end{aligned}$$

Example 7. Evaluate $\lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx}$.

Solution

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx} \quad \left[\frac{0}{0} \text{ form} \right] \\ &= \lim_{x \rightarrow 0} \left\{ \frac{\frac{\sin ax}{ax} \cdot ax + bx}{ax + \frac{\sin bx}{bx} \cdot bx} \right\} \\ &= \lim_{x \rightarrow 0} \frac{x \left\{ a \left(\frac{\sin ax}{ax} \right) + b \right\}}{x \left\{ a + b \left(\frac{\sin bx}{bx} \right) \right\}} \\ &= \frac{\lim_{ax \rightarrow 0} \left\{ a \left(\frac{\sin ax}{ax} \right) + b \right\}}{\lim_{bx \rightarrow 0} \left\{ a + b \left(\frac{\sin bx}{bx} \right) \right\}} \\ &= \frac{a(1) + b}{a + b(1)} = \frac{a+b}{a+b} \\ &= 1. \end{aligned}$$

Example 8. Evaluate: $\lim_{x \rightarrow y} \frac{\tan x - \tan y}{x - y}$.

Solution

$$\begin{aligned} & \lim_{x \rightarrow y} \frac{\tan x - \tan y}{x - y} \quad \left[\frac{0}{0} \text{ form} \right] \\ &= \lim_{x \rightarrow y} \frac{\frac{\sin x}{\cos x} - \frac{\sin y}{\cos y}}{x - y} \\ &= \lim_{x \rightarrow y} \frac{\sin x \cos y - \cos x \sin y}{(x - y) \cos x \cos y} \\ &= \lim_{x \rightarrow y} \frac{\sin(x - y)}{(x - y) \cos x \cos y} \\ &= \lim_{x - y \rightarrow 0} \frac{\sin(x - y)}{x - y} \lim_{x \rightarrow y} \frac{1}{\cos x \cos y} \\ &= 1 \cdot \frac{1}{\cos y \cos y} \left(\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right) \\ &= \sec^2 y. \end{aligned}$$

Example 9. Prove that $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{\sin x} = 1$.

Solution

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{\sin x} \quad \left[\frac{0}{0} \text{ form} \right] \\ &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - \sqrt{1-x})(\sqrt{1+x} + \sqrt{1-x})}{\sin x (\sqrt{1+x} + \sqrt{1-x})} \\ &= \lim_{x \rightarrow 0} \frac{(1+x) - (1-x)}{\sin x (\sqrt{1+x} + \sqrt{1-x})} \\ &= \lim_{x \rightarrow 0} \frac{2x}{\sin x (\sqrt{1+x} + \sqrt{1-x})} \\ &= 2 \lim_{x \rightarrow 0} \frac{x}{\sin x} \cdot \frac{1}{\sqrt{1+x} + \sqrt{1-x}} \\ &= 2 \times 1 \times \frac{1}{(\sqrt{1+0} + \sqrt{1-0})} \\ &= 2 \times \frac{1}{2} \\ &= 1. \end{aligned}$$

Example 10. Evaluate $\lim_{x \rightarrow 0} \frac{e^x - 1}{\sqrt{1 - \cos x}}$.

Solution

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{e^x - 1}{\sqrt{1 - \cos x}} \\ &= \lim_{x \rightarrow 0} \frac{e^x - 1}{\sqrt{2 \sin^2 \frac{x}{2}}} \\ &= \lim_{x \rightarrow 0} \frac{e^x - 1}{x} \cdot x \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sin \frac{x}{2}} \\ &= \lim_{x \rightarrow 0} \frac{e^x - 1}{x} \cdot \frac{1}{\sqrt{2}} \cdot \frac{\frac{x}{2} \cdot 2}{\sin \frac{x}{2}} \\ &= \sqrt{2} \lim_{x \rightarrow 0} \frac{e^x - 1}{x} \cdot \lim_{\frac{x}{2} \rightarrow 0} \frac{1}{\sin \frac{x}{2}} \\ &= \sqrt{2} \cdot 1 \cdot 1 \\ &= \sqrt{2}. \end{aligned}$$

Example 11. Evaluate $\lim_{\theta \rightarrow 0} \frac{x \cot \theta - \theta \cot x}{x - \theta}$.

Solution

When $x = \theta$, $\frac{x \cot \theta - \theta \cot x}{x - \theta}$ takes the indeterminate form $\frac{0}{0}$.

Put $x - \theta = h$ then $x = \theta + h$ when $x \rightarrow \theta$, $h \rightarrow 0$

Now,

$$\begin{aligned} & \lim_{x \rightarrow \theta} \frac{x \cot \theta - \theta \cot x}{x - \theta} \\ &= \lim_{h \rightarrow 0} \frac{(\theta + h) \cot \theta - \theta \cot (\theta + h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h \cot \theta + \theta [\cot \theta - \cot (\theta + h)]}{h} \\ &= \lim_{h \rightarrow 0} \left[\cot \theta + \frac{\theta}{h} \left\{ \frac{\cos \theta}{\sin \theta} - \frac{\cos (\theta + h)}{\sin (\theta + h)} \right\} \right] \\ &= \cot \theta + \lim_{h \rightarrow 0} \left[\frac{\theta}{h} \left\{ \frac{\cos \theta \sin (\theta + h) - \cos (\theta + h) \sin \theta}{\sin \theta \sin (\theta + h)} \right\} \right] \end{aligned}$$

$$\begin{aligned} &= \cot \theta + \lim_{h \rightarrow 0} \left[\frac{\theta}{h} \frac{\sin h}{\sin \theta \sin (\theta + h)} \right] \\ &= \cot \theta + \theta \lim_{h \rightarrow 0} \frac{\left(\frac{\sin h}{h} \right)}{\sin \theta \sin (\theta + h)} \\ &= \cot \theta + \theta \cdot \frac{1}{\sin \theta \cdot \sin \theta} \\ &= \cot \theta + \theta \operatorname{cosec}^2 \theta. \end{aligned}$$

Alternative method

$$\begin{aligned} & \lim_{x \rightarrow \theta} \frac{x \cot \theta - \theta \cot x}{x - \theta} \quad \left[\frac{0}{0} \text{ form} \right] \\ &= \lim_{x \rightarrow \theta} \left\{ \frac{x \cot \theta - \theta \cot \theta + \theta \cot \theta - \theta \cot x}{x - \theta} \right\} \\ &= \lim_{x \rightarrow \theta} \left\{ \frac{(x - \theta) \cot \theta + \theta (\cot \theta - \cot x)}{x - \theta} \right\} \\ &= \lim_{x \rightarrow \theta} \left\{ \frac{(x - \theta) \cot \theta}{(x - \theta)} + \frac{\theta}{(x - \theta)} \left(\frac{\cos \theta}{\sin \theta} - \frac{\cos x}{\sin x} \right) \right\} \\ &= \lim_{x \rightarrow \theta} \left\{ \cot \theta + \frac{\theta}{x - \theta} \left(\frac{\sin x \cos \theta - \cos x \sin \theta}{\sin \theta \sin x} \right) \right\} \\ &= \lim_{x \rightarrow \theta} \left\{ \cot \theta + \theta \cdot \frac{\sin (x - \theta)}{(x - \theta)} \cdot \frac{1}{\sin \theta \sin x} \right\} \\ &= \cot \theta + \theta \cdot 1 \cdot \frac{1}{\sin \theta \cdot \sin \theta} \\ &= \cot \theta + \theta \operatorname{cosec}^2 \theta. \end{aligned}$$

Example 12. Evaluate: $\lim_{x \rightarrow 0} \left(\frac{3^{2x} - 1}{2^{3x} - 1} \right)$.

Solution

$$\begin{aligned} & \lim_{x \rightarrow 0} \left(\frac{3^{2x} - 1}{2^{3x} - 1} \right) \quad \left[\frac{0}{0} \text{ form} \right] \\ &= \lim_{x \rightarrow 0} \frac{\left(\frac{3^{2x} - 1}{2x} \cdot 2x \right)}{\left(\frac{2^{3x} - 1}{3x} \cdot 3x \right)} = \frac{2}{3} \frac{\lim_{2x \rightarrow 0} \left(\frac{3^{2x} - 1}{2x} \right)}{\lim_{3x \rightarrow 0} \left(\frac{2^{3x} - 1}{3x} \right)} \\ &= \frac{2}{3} \frac{\log_e 3}{\log_e 2} \left[\because \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a \right] \\ &= \frac{\log_e 3}{\log_e 2} \\ &= \frac{\log_e 9}{\log_e 8}. \end{aligned}$$

Exercise 1.2

Evaluate the following limits (1 - 22):

1. $\lim_{x \rightarrow 0} \frac{\sin bx}{x}$
2. $\lim_{x \rightarrow 0} \frac{\tan 5x}{x}$
3. $\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 5x}$
4. $\lim_{x \rightarrow 0} \frac{\tan mx}{\tan nx}$
5. $\lim_{x \rightarrow 0} \frac{\tan ax}{\sin bx}$
6. $\lim_{x \rightarrow b} \frac{\tan(x-b)}{x^2 - b^2}$
7. $\lim_{x \rightarrow 0} \frac{\sin ax \cos bx}{\tan cx}$
8. $\lim_{x \rightarrow 0} \frac{\sin 2x + \sin 6x}{\sin 5x - \sin 3x}$
9. $\lim_{x \rightarrow 0} \frac{(\tan 3x - 2x)}{(3x - \sin 2x)}$
10. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$
11. $\lim_{x \rightarrow 0} \frac{1 - \cos 3x}{1 - \cos 5x}$
12. $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3}$
13. $\lim_{x \rightarrow \pi/4} \frac{\sec^2 x - 2}{\tan x - 1}$
14. $\lim_{x \rightarrow 0} \frac{\cos mx - \cos nx}{x^2}$
15. $\lim_{x \rightarrow y} \frac{\cos x - \cos y}{x - y}$
16. $\lim_{x \rightarrow y} \frac{\sin x - \sin y}{x - y}$
17. $\lim_{x \rightarrow \theta} \frac{x \tan \theta - \theta \tan x}{x - \theta}$
18. $\lim_{x \rightarrow \theta} \frac{x \sin \theta - \theta \sin x}{x - \theta}$
19. $\lim_{x \rightarrow \theta} \frac{x \cos \theta - \theta \cos x}{x - \theta}$
20. $\lim_{y \rightarrow 0} \frac{(x+y) \sec(x+y) - x \sec x}{y}$
21. $\lim_{x \rightarrow \pi/4} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$
22. $\lim_{x \rightarrow c} \frac{\sin x - \sin c}{\sqrt{x} - \sqrt{c}}$

Evaluate (23 - 29)

23. $\lim_{x \rightarrow 0} \frac{e^x - 1}{3x}$
24. $\lim_{x \rightarrow 0} \frac{e^{5x} - 1}{x \cdot 3^x}$
25. $\lim_{x \rightarrow 0} \frac{e^{2+x} - e^2}{x}$
26. $\lim_{x \rightarrow 0} \frac{\log_e(1+x)}{e^x - 1}$
27. $\lim_{x \rightarrow 0} \frac{\log_e(1+5x)}{2x}$
28. $\lim_{x \rightarrow 0} \frac{a^x - b^x}{x}$
29. $\lim_{x \rightarrow 0} \frac{3^{2x} - 2^{3x}}{x}$

Very Short Answer Questions

1. In $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$, in which measurement system the angle θ is measured?
2. If $\lim_{x \rightarrow 0} \frac{\sin kx}{x} = 2$ then find the value of k .
3. Evaluate:
 - a. $\lim_{x \rightarrow 0} \cos x$
 - b. $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta}$
 - c. $\lim_{x \rightarrow 0} \frac{\sin x^\circ}{x}$
 - d. $\lim_{x \rightarrow 0} \frac{\sin x^e}{x}$
 - e. $\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}$
 - f. $\lim_{x \rightarrow 0} \frac{\tan \alpha x}{\sin \beta x}$
 - g. $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2}$
 - h. $\lim_{x \rightarrow 0} \sin \frac{1}{x}$
 - i. $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x}$
 - j. $\lim_{x \rightarrow 0} \frac{3^x - 1}{x}$
 - k. $\lim_{x \rightarrow 1} \frac{\log x}{x - 1}$
 - l. $\lim_{x \rightarrow \infty} \frac{\sin x}{x}$

Answers

1. b
2. 5
3. $\frac{4}{5}$
4. $\frac{m}{n}$
5. $\frac{a}{b}$
6. $\frac{1}{2b}$
7. $\frac{a}{c}$
8. 4
9. 1
10. $\frac{1}{2}$
11. $\frac{9}{25}$
12. $\frac{1}{2}$
13. 2
14. $\frac{1}{2}(n^2 - m^2)$
15. $-\sin y$
16. $\cos y$
17. $\tan \theta - \theta \sec^2 \theta$
18. $\sin \theta - \theta \cos \theta$
19. $\cos \theta + \theta \sin \theta$
20. $\sec x(1 + x \tan x)$
21. $-\sqrt{2}$
22. $2\sqrt{c} \cos c$
23. $\frac{1}{3}$
24. 5
25. e^2
26. 1
27. $\frac{5}{2}$
28. $\log_e \left(\frac{a}{b} \right)$
29. $\log_e \left(\frac{9}{8} \right)$

Very Short Answer Questions

2. 2
3. a. 1
c. $\frac{a}{b}$
e. $\frac{a}{b}$
i. 2
- b. 1
f. $\frac{\alpha}{\beta}$
j. $\log_e 3$
- c. $\frac{\pi}{180}$
g. 2
k. 1
- d. $\frac{\pi}{200}$
h. Doesn't exist
l. 0

1.10 Limits

Neighborhood

Any open interval (a, b) containing $c \in \mathbb{R}$ is called a neighborhood of c .

Example: $(0, 1)$ is neighborhood of 0.3, 0.4, 0.5, ..., 0.9 etc.

Thus a neighborhood of a point a is an open interval that contains the point a . It is denoted by $(a - \delta, a + \delta)$ where δ (delta) > 0 .

Let $x \in (a - \delta, a + \delta)$

$$\Rightarrow a - \delta < x < a + \delta$$

$$\Rightarrow -\delta < x - a < \delta$$

$$\Rightarrow |x - a| < \delta$$

Thus $x \in (a - \delta, a + \delta)$ is equivalent to $|x - a| < \delta$.

As we have discussed already, $\lim_{x \rightarrow a} f(x) = \ell$ means $f(x)$ approaches ℓ when x approaches a . i.e. $|f(x) - \ell|$ is very small whenever $|x - a|$ is very small. In other words we can make $|f(x) - \ell|$ as small as we please by choosing x such that $|x - a|$ is sufficiently small. This concept gives rise to the following definition of limit.

1.10.1 Definition of Limit ($\epsilon - \delta$ Definition)

Let $f(x)$ be defined in a neighborhood of a [$f(x)$ may or may not be defined at $x = a$]. Then we say $\lim_{x \rightarrow a} f(x) = \ell$ if for every $\epsilon > 0$, however small, there

corresponds a number $\delta > 0$, such that

$$|f(x) - \ell| < \epsilon \text{ whenever } |x - a| < \delta.$$

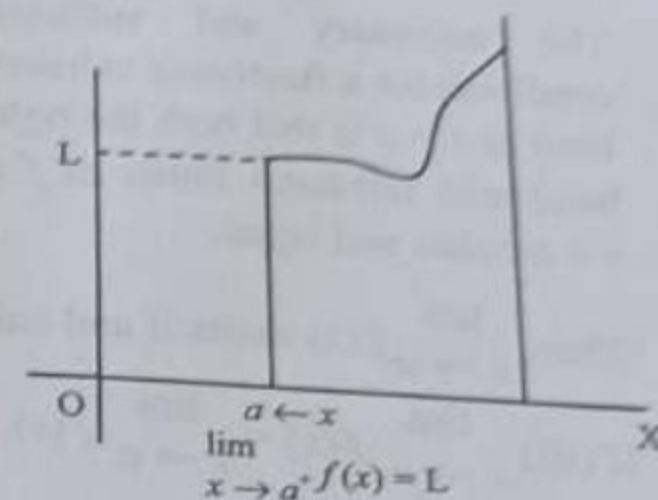
One Sided-limit

1. **Right-hand limit:** Consider the right hand neighborhood $(a, a + \delta)$ of a . Then L is said to be the right hand limit of f at $x = a$ if given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever

$x \in (a, a + \delta)$. It is denoted by

$$\lim_{x \rightarrow a+0} f(x) \text{ or } \lim_{x \rightarrow a^+} f(x)$$

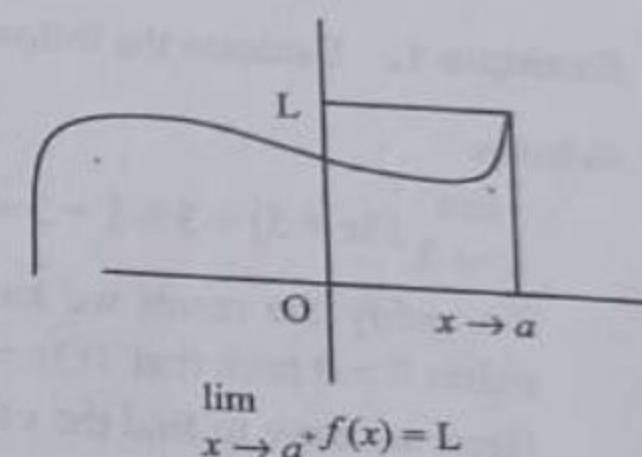
$$\text{or } f(a+) \text{ or } \lim_{h \rightarrow 0} f(a+h).$$



2. **Left hand limit:** Consider the left hand neighborhood $(a - \delta, a)$ of a . Then L is called the left hand limit of f at $x = a$ if given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \in (a - \delta, a)$. It is denoted by

$$\lim_{x \rightarrow a-0} f(x) \text{ or } \lim_{x \rightarrow a^-} f(x) \text{ or }$$

$$f(a-) \text{ or } \lim_{h \rightarrow 0} f(a-h).$$



One Sided Infinite Limits

Example 1. Evaluate $\lim_{x \rightarrow 0^+} \frac{1}{x}$ and $\lim_{x \rightarrow 0^-} \frac{1}{x}$.

Solution

As x approaches zero from right side, the value of $\frac{1}{x}$ are positive and becomes arbitrarily large. So

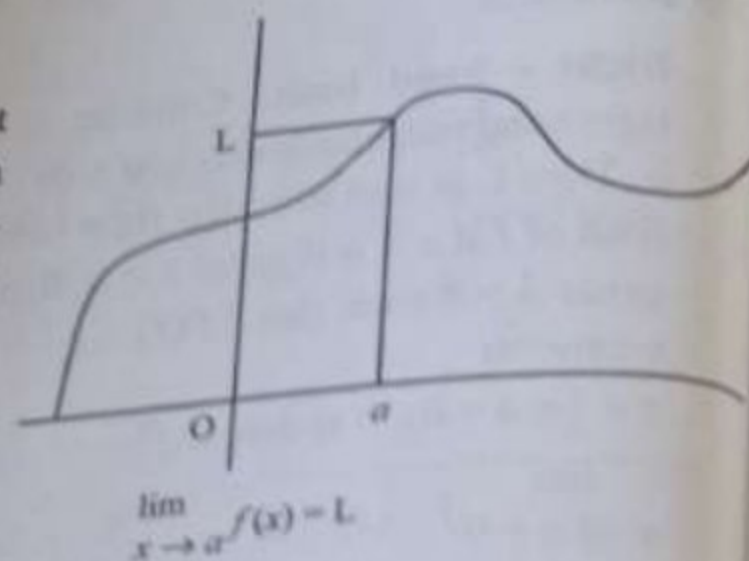
$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$$

Again, as x approaches zero from left side, the value of $\frac{1}{x}$ are negative and becomes arbitrarily small. So, $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$.

Two Sided-limits

The necessary and sufficient condition for a function f to have a limit at $x = a$ is that both the right-hand and left-hand limits of f at $x = a$ exist and are equal.

Thus, $\lim_{x \rightarrow a} f(x)$ exists if and only if (iff) $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$.



Illustrative Examples

Example 1. Evaluate the following limit and verify $\lim_{x \rightarrow 1} (3x + 5)$.

Solution

$$\lim_{x \rightarrow 1} (3x + 5) = 3 \times 1 + 5 = 8$$

To verify the result we have to show that corresponding to $\epsilon > 0$, there exists $\delta > 0$ such that $|(3x + 5) - 8| < \epsilon$ whenever $|x - 1| < \delta$.

Here, we have to find the value of δ in terms of ϵ considering $\epsilon > 0$ is given.

$$\text{Now, } |(3x + 5) - 8| < \epsilon$$

$$\text{Whenever } |3x - 3| < \epsilon$$

$$|x - 1| < \frac{\epsilon}{3}$$

$$\text{Thus } \delta = \frac{\epsilon}{3}.$$

Example 2. Calculate the limits at point specified $f(x) = \begin{cases} 3x + 2 & \text{for } x \geq 0 \\ x + 2 & \text{for } x < 0 \end{cases}$ at $x = 0$.

Solution

Left hand limit at $x = 0$ is

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} (x + 2) \\ &= 0 + 2 \\ &= 2 \end{aligned}$$

Right hand limit at $x = 0$ is

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} (3x + 2) \\ &= 3 \times 0 + 2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \therefore \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^+} f(x) = 2 \\ \text{Thus, } \lim_{x \rightarrow 0} f(x) &= 2. \end{aligned}$$

Example 3. Evaluate: $\lim_{x \rightarrow 0} \frac{|x|}{x}$ if it exists.

Solution

$$\text{We have } |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\text{Let } f(x) = \frac{|x|}{x}$$

Now, right hand limit at $x = 0$ is

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \frac{|x|}{x} \\ &= \lim_{x \rightarrow 0^+} \frac{x}{x} \\ &= \lim_{x \rightarrow 0^+} (1) \\ &= 1 \end{aligned}$$

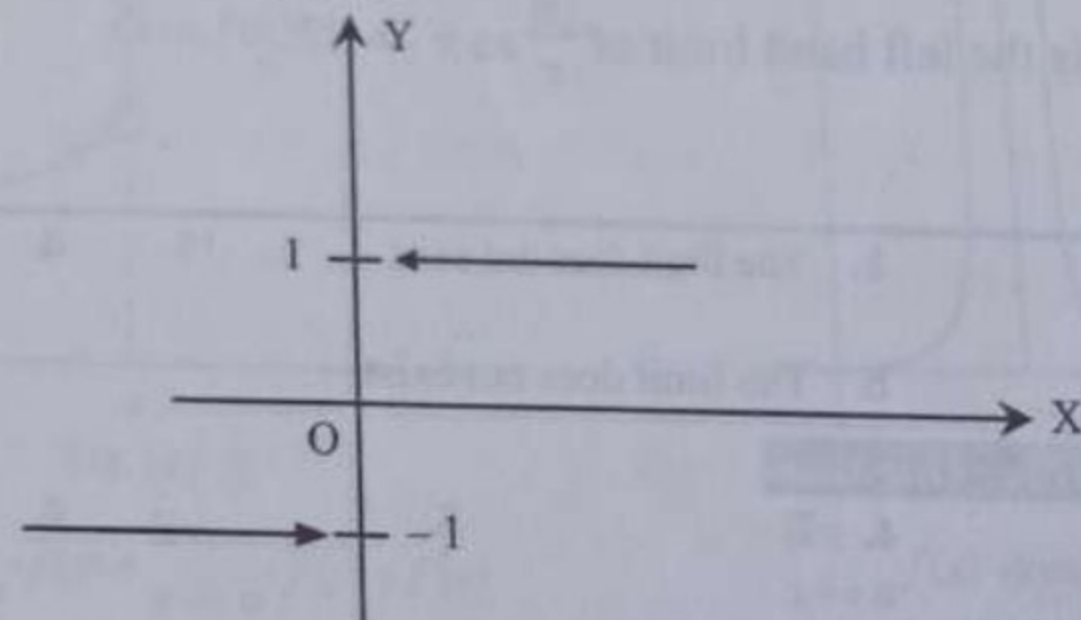
Left hand limit at $x = 0$ is

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \frac{|x|}{x} \\ &= \lim_{x \rightarrow 0^-} \frac{-x}{x} \\ &= \lim_{x \rightarrow 0^-} (-1) \\ &= -1 \end{aligned}$$

$$\text{Thus } \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x).$$

So, $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist.

The graph is shown along side.



Exercise 1.3

1. Find the limits at the points specified.

a. $f(x) = \begin{cases} 4x+2 & \text{for } x \geq 0 \\ 2 & \text{for } x < 0 \end{cases}$ at $x=0$

b. $f(x) = \begin{cases} 2x-3 & \text{for } x \geq 1 \\ x & \text{for } x < 1 \end{cases}$ at $x=1$

c. $f(x) = \begin{cases} 7x+1 & \text{for } x \geq 2 \\ x^2+11 & \text{for } x < 2 \end{cases}$ at $x=2$

d. $f(x) = \begin{cases} 3x-1 & \text{for } x < 1 \\ 5 & \text{for } x = 1 \\ 2x & \text{for } x > 1 \end{cases}$ at $x=1$

2. Let $f(x) = \begin{cases} 7x+1 & \text{for } x \geq 2 \\ k & \text{for } x < 2 \end{cases}$. If $\lim_{x \rightarrow 2} f(x)$ exists then find the value of k .

3. Evaluate the following limits:

a. $\lim_{x \rightarrow 3} |x-3|$

b. $\lim_{x \rightarrow 0} \frac{x}{|x|}$

Very Short Answer Questions

1. Define $\lim_{x \rightarrow a^-} f(x)$.

2. Define $\lim_{x \rightarrow a^+} f(x)$.

3. A function $f(x)$ is defined by $f(x) = \begin{cases} 2x+1 & x \geq 1 \\ x+4 & x < 1 \end{cases}$ then find $\lim_{x \rightarrow 1^+} f(x)$

4. A function $f(x)$ is defined by $f(x) = \begin{cases} 2x-3 & \text{for } x \geq 3 \\ 3x-2 & \text{for } x < 3 \end{cases}$ then find $\lim_{x \rightarrow 3^-} f(x)$

5. If $f(x) = \begin{cases} x+1 & \text{if } x \geq 1 \\ 2x & \text{if } x < 1 \end{cases}$ then find $\lim_{x \rightarrow 1} f(x)$.

6. What is the left hand limit of $\frac{|x|}{x}$ as $x \rightarrow 0$?

Answers

1. a. 2 b. The limit does not exist c. 15 d. 2
2. 15
3. a. 0 b. The limit does not exist

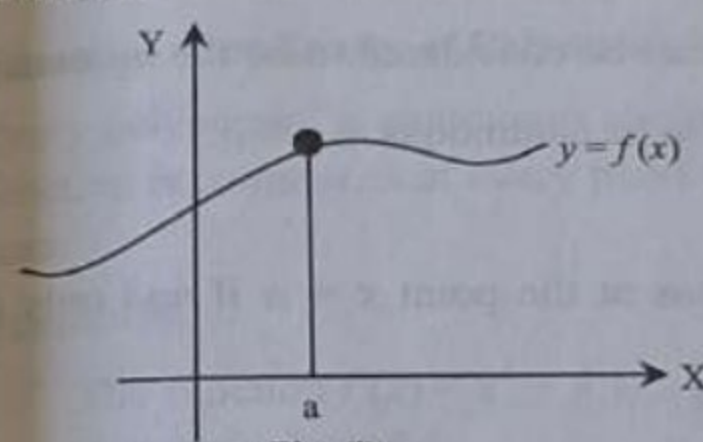
Very Short Answer Questions

3. 3 4. 7 5. 2 6. -1

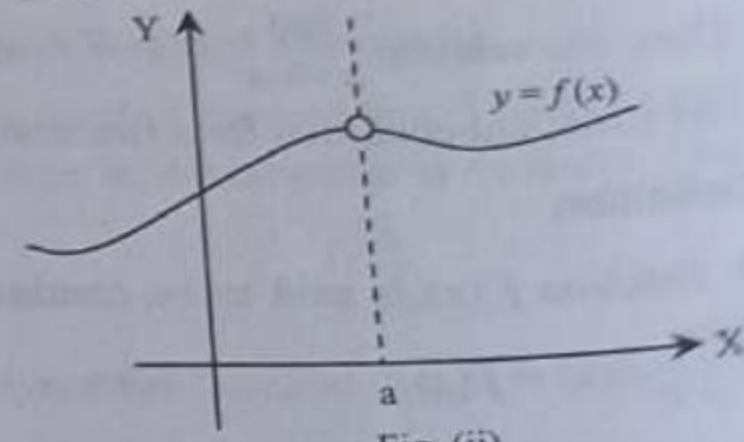
1.11 Continuity

Introduction

A function whose graph can be drawn without lifting the pencil in a sheet of paper is continuous or smooth. Mathematically, this situation arises if the function is defined at a given point and its limit at the given point exists and is equal to the value of function at that point. Otherwise the function is discontinuous. Pictorially, graphs of discontinuous functions consist of disconnected curves. Let us look at their graphs and discuss their nature.

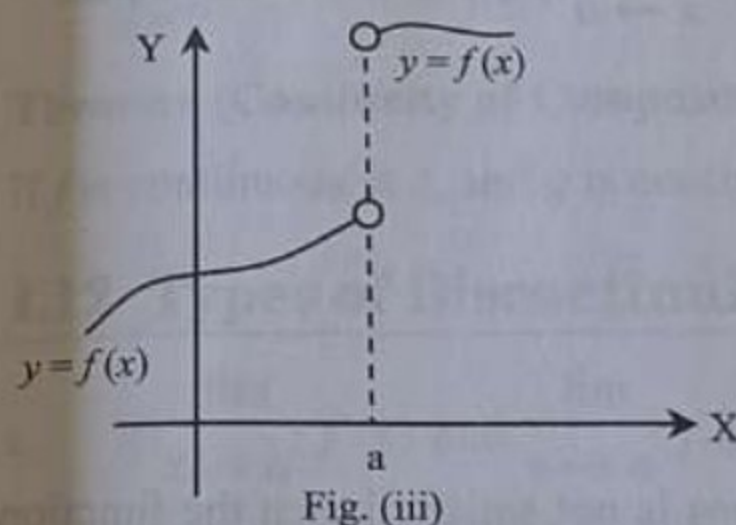


$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$



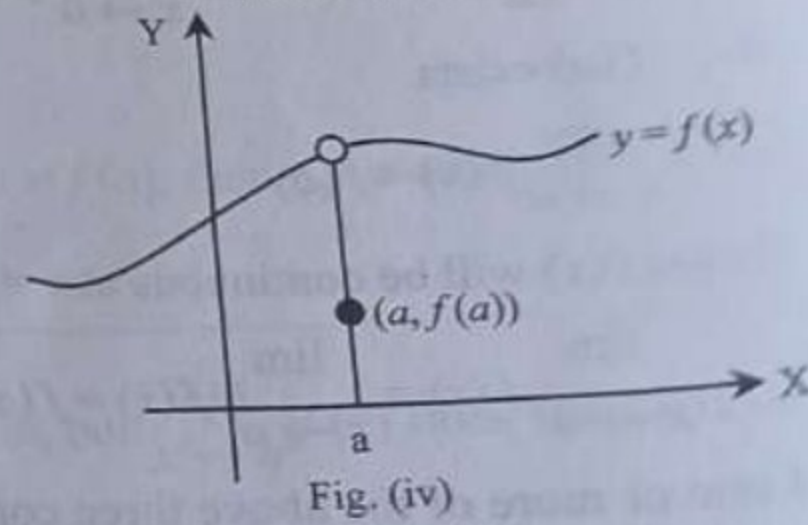
$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$$

But $f(a)$ is not defined.

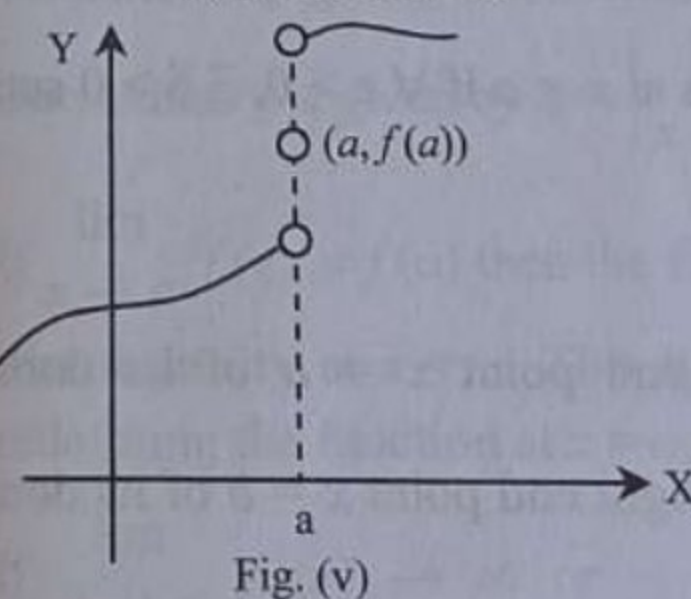


$$\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$$

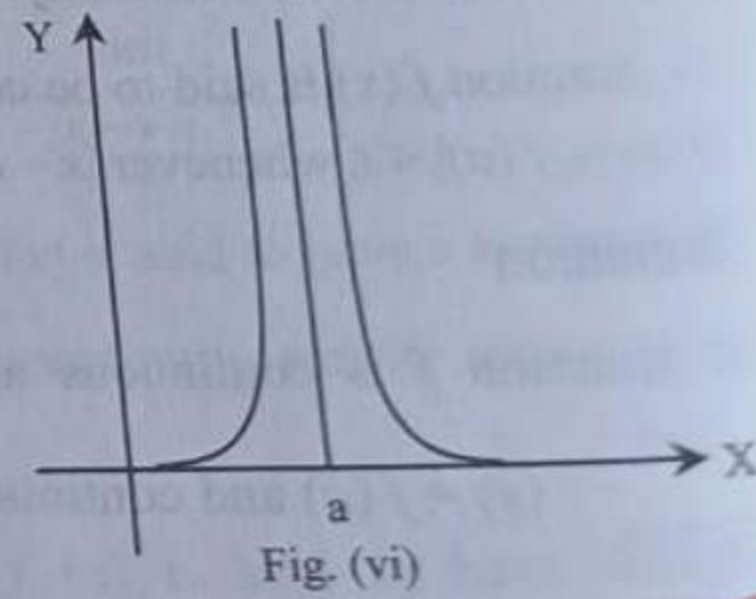
$f(a)$ is not defined



$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) \neq f(a)$$



$$\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x) \neq f(a)$$



$$\lim_{x \rightarrow a} f(x) \text{ does not exist and } f \text{ not defined at } x=a$$

Now, it is clearly seen that except the graph of the function f in fig. (i), all other graphs fig. (ii) to fig. (vi) are discontinuous. This may happen in various ways. Such situation arises if

- i. $f(a)$ is not defined.
- ii. $\lim_{x \rightarrow a} f(x)$ does not exist.
- iii. $\lim_{x \rightarrow a} f(x) \neq f(a)$

Thus, the relation $\lim_{x \rightarrow a} f(x) = f(a)$ can be considered to be the necessary and sufficient condition for a function f to be continuous at $x = a$.

Definition

A function $f(x)$ is said to be continuous at the point $x = a$ if and only if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

Equivalently, the function $f(x)$ is said to be continuous if

- a. $\lim_{x \rightarrow a} f(x)$ exists i.e. $\lim_{x \rightarrow a} {}^-f(x)$ and $\lim_{x \rightarrow a} {}^+f(x)$ are finite and equal.
- b. $f(a)$ exists.
- c. $\lim_{x \rightarrow a} f(x) = f(a)$

Hence $f(x)$ will be continuous at $x = a$ iff

$$\lim_{x \rightarrow a} {}^-f(x) = \lim_{x \rightarrow a} {}^+f(x) = f(a)$$

If one or more of the above three conditions is not satisfied then the function $f(x)$ is said to be discontinuous at the point $x = a$.

$\epsilon - \delta$ Definition of Continuity

A function $f(x)$ is said to be continuous at $x = a$ if $\forall \epsilon > 0, \exists \delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Definition

A function f is continuous at a left end point $x = a$ of its domain if

$$\lim_{x \rightarrow a} {}^+f(x) = f(a) \text{ and continuous at a right end point } x = b \text{ of its domain if}$$

$$\lim_{x \rightarrow a} {}^-f(x) = f(b).$$

Theorem (Continuity of Algebraic Combinations)

If functions f and g are continuous at $x = a$, then the following functions are continuous at $x = a$.

1. $f + g$ and $f - g$
2. fg
3. kf , where k is any number
4. $\frac{f}{g}, g(a) \neq 0$.

Theorem (Continuity of Polynomials and Rational Functions)

Every polynomial is continuous at every point of the real line. Every rational function is continuous at every point where its denominator is different from zero.

Examples:

- i. The function $f(x) = x^3 + 5$ is a polynomial function. So, it is continuous at every value of x .
- ii. The function $f(x) = \frac{x+3}{x-2}$ is continuous at every value of x except $x = 2$.

Theorem (Continuity of Composites)

If f is continuous at a , and g is continuous at $f(a)$, then $g \circ f$ is continuous at a .

1.12 Types of Discontinuities

- i. If $\lim_{x \rightarrow a} {}^+f(x)$ and $\lim_{x \rightarrow a} {}^-f(x)$ exist but $\lim_{x \rightarrow a} f(x)$ does not exist i.e.

$\lim_{x \rightarrow a} {}^-f(x) \neq \lim_{x \rightarrow a} {}^+f(x)$ then $f(x)$ is said to have an **ordinary discontinuity (jump discontinuity)** at $x = a$. The height or jump of this

discontinuity is given by $h = \left| \lim_{x \rightarrow a} {}^+f(x) - \lim_{x \rightarrow a} {}^-f(x) \right|$.

- ii. If $\lim_{x \rightarrow a} f(x) \neq f(a)$ then the function $f(x)$ is said to have a **removable discontinuity** at $x = a$. This type of discontinuity can be removed by redefining the function at $x = a$.

- iii. If $\lim_{x \rightarrow a} f(x) \rightarrow \infty$ or $-\infty$ then $f(x)$ is said to have **infinite discontinuity** at $x = a$.

Illustrative Examples

Example 1. Test the continuity or discontinuity of the function $f(x) = 3x^2 + 5x + 4$ at $x = 1$.

Solution

Left hand limit at $x = 1$ is

$$\begin{aligned}\lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} (3x^2 + 5x + 4) \\ &= 3 \cdot 1^2 + 5 \cdot 1 + 4 \\ &= 12\end{aligned}$$

Right hand limit at $x = 1$ is

$$\begin{aligned}\lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} (3x^2 + 5x + 4) \\ &= 3 \cdot 1^2 + 5 \cdot 1 + 4 \\ &= 12\end{aligned}$$

Hence, $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$ are finite and equal.

$$\text{So, } \lim_{x \rightarrow 1} f(x) \text{ exists and } \lim_{x \rightarrow 1} f(x) = 12$$

Also,

$$f(1) = 3 \cdot 1^2 + 5 \cdot 1 + 4 = 12$$

$$\therefore \lim_{x \rightarrow 1} f(x) = f(1)$$

Hence $f(x)$ is continuous at $x = 1$.

Example 2. Is the function $f(x) = \frac{x^2 - 4}{x - 2}$ continuous at $x = 2$?

Solution

Left hand limit at $x = 2$ is

$$\begin{aligned}\lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \rightarrow 2^-} \frac{(x + 2)(x - 2)}{(x - 2)} \\ &= \lim_{x \rightarrow 2^-} (x + 2) \\ &= 2 + 2 \\ &= 4\end{aligned}$$

Right hand limit at $x = 2$ is

$$\begin{aligned}\lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \rightarrow 2^+} \frac{(x + 2)(x - 2)}{(x - 2)} \\ &= \lim_{x \rightarrow 2^+} (x + 2) \\ &= 2 + 2 \\ &= 4\end{aligned}$$

And $f(2) = \frac{x^2 - 4}{x - 2} = \frac{0}{0}$ which is an indeterminate form

So, $\lim_{x \rightarrow 2} f(x) \neq f(2)$ and hence the function $f(x) = \frac{x^2 - 4}{x - 2}$ is not continuous at $x = 2$.

Example 3. Show that the function $f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{for } x \neq 3 \\ 6 & \text{for } x = 3 \end{cases}$ is continuous at $x = 3$.

Solution

Left hand limit at $x = 3$ is

$$\begin{aligned}\lim_{x \rightarrow 3^-} f(x) &= \lim_{x \rightarrow 3^-} \frac{x^2 - 9}{x - 3} \\ &= \lim_{x \rightarrow 3^-} \frac{(x + 3)(x - 3)}{(x - 3)} \\ &= \lim_{x \rightarrow 3^-} (x + 3) \\ &= 3 + 3 \\ &= 6\end{aligned}$$

Right hand limit at $x = 3$ is

$$\begin{aligned}\lim_{x \rightarrow 3^+} f(x) &= \lim_{x \rightarrow 3^+} \frac{x^2 - 9}{x - 3} \\ &= \lim_{x \rightarrow 3^+} \frac{(x + 3)(x - 3)}{(x - 3)} \\ &= \lim_{x \rightarrow 3^+} (x + 3) \\ &= 3 + 3 \\ &= 6\end{aligned}$$

$\therefore \lim_{x \rightarrow 3^-} f(x)$ and $\lim_{x \rightarrow 3^+} f(x)$ both exist and are equal.

$$\text{Thus } \lim_{x \rightarrow 3} f(x) = 6$$

Illustrative Examples

Example 1. Test the continuity or discontinuity of the function $f(x) = 3x^2 + 5x + 4$ at $x = 1$.

Solution

Left hand limit at $x = 1$ is

$$\begin{aligned}\lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} (3x^2 + 5x + 4) \\ &= 3 \cdot 1^2 + 5 \cdot 1 + 4 \\ &= 12\end{aligned}$$

Right hand limit at $x = 1$ is

$$\begin{aligned}\lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} (3x^2 + 5x + 4) \\ &= 3 \cdot 1^2 + 5 \cdot 1 + 4 \\ &= 12\end{aligned}$$

Hence, $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$ are finite and equal.

So, $\lim_{x \rightarrow 1} f(x)$ exists and $\lim_{x \rightarrow 1} f(x) = 12$

Also,

$$f(1) = 3 \cdot 1^2 + 5 \cdot 1 + 4 = 12$$

$$\therefore \lim_{x \rightarrow 1} f(x) = f(1)$$

Hence $f(x)$ is continuous at $x = 1$.

Example 2. Is the function $f(x) = \frac{x^2 - 4}{x - 2}$ continuous at $x = 2$?

Solution

Left hand limit at $x = 2$ is

$$\begin{aligned}\lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \rightarrow 2^-} \frac{(x + 2)(x - 2)}{(x - 2)} \\ &= \lim_{x \rightarrow 2^-} (x + 2) \\ &= 2 + 2 \\ &= 4\end{aligned}$$

Right hand limit at $x = 2$ is

$$\begin{aligned}\lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \rightarrow 2^+} \frac{(x + 2)(x - 2)}{(x - 2)} \\ &= \lim_{x \rightarrow 2^+} (x + 2) \\ &= 2 + 2 \\ &= 4\end{aligned}$$

And $f(2) = \frac{x^2 - 4}{x - 2} = \frac{0}{0}$ which is an indeterminate form

So, $\lim_{x \rightarrow 2} f(x) \neq f(2)$ and hence the function $f(x) = \frac{x^2 - 4}{x - 2}$ is not continuous at $x = 2$.

Example 3. Show that the function $f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{for } x \neq 3 \\ 6 & \text{for } x = 3 \end{cases}$ is continuous at $x = 3$.

Solution

Left hand limit at $x = 3$ is

$$\begin{aligned}\lim_{x \rightarrow 3^-} f(x) &= \lim_{x \rightarrow 3^-} \frac{x^2 - 9}{x - 3} \\ &= \lim_{x \rightarrow 3^-} \frac{(x + 3)(x - 3)}{(x - 3)} \\ &= \lim_{x \rightarrow 3^-} (x + 3) \\ &= 3 + 3 \\ &= 6\end{aligned}$$

Right hand limit at $x = 3$ is

$$\begin{aligned}\lim_{x \rightarrow 3^+} f(x) &= \lim_{x \rightarrow 3^+} \frac{x^2 - 9}{x - 3} \\ &= \lim_{x \rightarrow 3^+} \frac{(x + 3)(x - 3)}{(x - 3)} \\ &= \lim_{x \rightarrow 3^+} (x + 3) \\ &= 3 + 3 \\ &= 6\end{aligned}$$

$\therefore \lim_{x \rightarrow 3^-} f(x)$ and $\lim_{x \rightarrow 3^+} f(x)$ both exist and are equal.

Thus $\lim_{x \rightarrow 3} f(x) = 6$

Also, when $x = 3$, $f(3) = 6$

$$\lim_{x \rightarrow 3} f(x) = f(3)$$

Hence the given function $f(x)$ is continuous at $x = 3$.

Example 4. Test the continuity of the function $f(x) = |x|$ at $x = 0$.

Solution

$$\text{We have, } |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\therefore f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\text{The left hand limit of } f(x) \text{ at } x = 0 \text{ is } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (-x) = 0$$

$$\text{The right hand limit of } f(x) \text{ at } x = 0 \text{ is } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x) = 0$$

The functional value of $f(x)$ at $x = 0$ is $f(0) = 0$.

$$\text{Therefore, } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0) = 0$$

So, the function $f(x) = |x|$ is continuous at $x = 0$.

Example 5. Discuss the continuity of the function

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases} \text{ at } x = 0.$$

Solution

We know that for all real x , $-1 \leq \sin \frac{1}{x} \leq 1$.

If $x > 0$, we have $-x \leq x \sin \frac{1}{x} \leq x$.

Taking $\lim_{x \rightarrow 0^+}$ on each of the terms, we get,

$$0 \leq \lim_{x \rightarrow 0^+} x \sin \frac{1}{x} \leq 0.$$

$$\text{Hence, } \lim_{x \rightarrow 0^+} x \sin \frac{1}{x} = 0$$

Similarly, if $x < 0$, we have $-x \geq x \sin \frac{1}{x} \geq x$.

Taking $\lim_{x \rightarrow 0^-}$ on each of the terms, we get,

$$0 \geq \lim_{x \rightarrow 0^-} x \sin \frac{1}{x} \geq 0.$$

$$\text{Hence, } \lim_{x \rightarrow 0^-} x \sin \frac{1}{x} = 0.$$

Also, $f(0) = 0$.

Since left hand limit = right hand limit = functional value,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0) = 0,$$

$\therefore f(x)$ is continuous at $x = 0$.

Example 6. For what value of k is the following function continuous at $x = 1$?

$$f(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ k & x = 1 \end{cases}$$

Solution

$$\begin{aligned} \lim_{x \rightarrow 1} f(x) &= \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{(x + 1)(x - 1)}{(x - 1)} \\ &= \lim_{x \rightarrow 1} (x + 1) \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

And $f(1) = k$

Since $f(x)$ is continuous at $x = 1$, so

$$\lim_{x \rightarrow 1} f(x) = f(1)$$

$$\therefore k = 2.$$

Example 7. If a function $f(x)$ is defined as $f(x) = \begin{cases} 3x^2 + 2 & \text{if } x < 1 \\ 2x + 3 & \text{if } x > 1 \\ 4 & \text{if } x = 1 \end{cases}$.

Discuss the continuity of function at $x = 1$.

Solution

Left hand limit at $x = 1$ is

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} (3x^2 + 2) \\ &= 3 \times 1^2 + 2 \\ &= 5 \end{aligned}$$

Right hand limit at $x = 1$ is

$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} (2x + 3) \\ &= 2 \times 1 + 3 \\ &= 5 \end{aligned}$$

Here $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$ are finite and equal.

So, $\lim_{x \rightarrow 1} f(x)$ exists and $\lim_{x \rightarrow 1} f(x) = 5$.

$$\text{And, } f(1) = 4.$$

$$\therefore \lim_{x \rightarrow 1} f(x) \neq f(1).$$

Hence $f(x)$ is not continuous at $x = 1$.

Example 8. Let a function $f(x)$ be defined by $f(x) = \begin{cases} 3x-1 & \text{for } x < 1 \\ 3 & \text{for } x = 1 \\ 2x & \text{for } x > 1 \end{cases}$

Verify that the limit of the function $f(x)$ exists at $x = 1$. Is the function $f(x)$ continuous at $x = 1$? If not, state how can you make it continuous at $x = 1$.

Solution

Left hand limit at $x = 1$ is

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} (3x-1) \\ &= 3 \times 1 - 1 \\ &= 2 \end{aligned}$$

Right hand limit at $x = 1$ is

$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} 2x \\ &= 2 \times 1 \\ &= 2 \end{aligned}$$

So, the limit of the function $f(x)$ at $x = 1$ exists.

Now, when $x = 1$, $f(1) = 3$.

$$\text{So, } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) \neq f(1)$$

Hence, the function is not continuous at $x = 1$.

This is a case for removable discontinuity. The function can be made continuous by redefining as follows: $f(x) = \begin{cases} 3x-1 & \text{for } x < 1 \\ 2 & \text{for } x = 1 \\ 2x & \text{for } x > 1 \end{cases}$

Exercise 1.4

1. Test the continuity or discontinuity of the following functions at the points specified.

a. $f(x) = x^3 + 3x + 2$ at $x = 1$

b. $f(x) = 7 - x^2$ at $x = 0$

c. $f(x) = \frac{1}{3x}$ at $x = 0$

d. $f(x) = \frac{1}{1-x}$ at $x = 1$

e. $f(x) = \frac{1}{x-3}$ at $x \neq 3$

f. $f(x) = \frac{x^2-4}{x-2}$ at $x = 2$

g. $f(x) = \frac{|x-1|}{x-1}$ at $x = 1$.

2. Find the points of discontinuity of the following functions:

a. $f(x) = \frac{x+5}{x-4}$

b. $f(x) = \frac{x^2}{x^2-3x+2}$

3. Are the following functions continuous at the points mentioned?

a. $f(x) = \begin{cases} x^2-4 & \text{when } x \neq 2 \\ 4 & \text{when } x = 2 \end{cases}$ at $x = 2$

b. $f(x) = \begin{cases} x^2-3x & x \neq 3 \\ 3 & x = 3 \end{cases}$ at $x = 3$

c. $f(x) = \begin{cases} x^2-x-6 & \text{if } x \neq 3 \\ 3 & \text{if } x = 3 \end{cases}$ at $x = 3$

4. Discuss the continuity of the function at the points specified.

a. $f(x) = \begin{cases} 2-x^2 & \text{for } x \leq 1 \\ x & \text{for } x > 1 \end{cases}$ at $x = 1$

b. $f(x) = \begin{cases} 3x^2+5 & \text{for } x \geq 2 \\ 2x+11 & \text{for } x < 2 \end{cases}$ at $x = 2$

c. $f(x) = \begin{cases} 2x-1 & \text{for } x < 2 \\ 3 & \text{for } x = 2 \\ x+1 & \text{for } x > 2 \end{cases}$ at $x = 2$

d. $f(x) = \begin{cases} 3+2x & \text{for } -\frac{3}{2} \leq x < 0 \\ 3-2x & \text{for } 0 \leq x < \frac{3}{2} \\ -3-2x & \text{for } x \geq \frac{3}{2} \end{cases}$ (i) at $x = 0$ (ii) at $x = \frac{3}{2}$

5. Determine the value of the constant so that the given function is continuous at the point mentioned.

a. $f(x) = \begin{cases} x^2-9 & \text{if } x \neq 3 \\ k & \text{if } x = 3 \end{cases}$ at $x = 3$

b. $f(x) = \begin{cases} 2x-1 & \text{if } x < 2 \\ k & \text{if } x = 2 \\ x+1 & \text{if } x > 2 \end{cases}$ at $x = 2$

c. $f(x) = \begin{cases} 2ax+3 & \text{if } x < 1 \\ 1-ax^2 & \text{if } x \geq 1 \end{cases}$ at $x = 1$

6. a. A function $f(x)$ is defined as follows

$$f(x) = \begin{cases} 2x+3 & \text{for } x < 1 \\ 4 & \text{for } x = 1 \\ 6x-1 & \text{for } x > 1 \end{cases}$$

Is the function $f(x)$ continuous at $x = 1$? If not, state how can you make it continuous at $x = 1$.

b. Let a function $f(x)$ be defined by

$$f(x) = \begin{cases} 2-x^2 & \text{for } x < 2 \\ 3 & \text{for } x = 2 \\ x-4 & \text{for } x > 2 \end{cases}$$

Verify that the limit of the function exists at $x = 2$. Is the function continuous at $x = 2$? State how you can make it continuous.

Very Short Answer Questions

- Write any two types of discontinuity.
- When a function $f(x)$ is said to have removable discontinuity at $x = a$?
- When a function f is continuous at a left end point $x = a$ of its domain?
- Find the point of discontinuity of $y = \frac{x^2-4}{x-2}$.
- Write the type of discontinuity of the function $f(x) = \sin \frac{1}{x}$ at $x = 0$.

6. If the function $f(x) = \begin{cases} \frac{x^2-9}{x-3} & x \neq 3 \\ k & x = 3 \end{cases}$ is continuous at $x = 3$ then find the value of k .

7. If the function $f(x) = \begin{cases} 2x-1 & x < 2 \\ a & x = 2 \\ x+1 & x > 2 \end{cases}$ is continuous at $x = 2$ then find a .

Answers

- a. continuous b. continuous c. discontinuous d. discontinuous
- e. continuous f. discontinuous g. discontinuous
- a. $x = 4$ b. $x = 1, 2$
- a. continuous b. continuous c. discontinuous
- a. continuous b. discontinuous c. continuous
- d. i. continuous ii. discontinuous
- a. 6 b. 3 c. $-\frac{2}{3}$

6. a. No, $f(x) = \begin{cases} 2x+3 & \text{for } x < 1 \\ 5 & \text{for } x = 1 \\ 6x-1 & \text{for } x > 1 \end{cases}$

b. No, $f(x) = \begin{cases} 2-x^2 & \text{for } x < 2 \\ -2 & \text{for } x = 2 \\ x-4 & \text{for } x > 2 \end{cases}$

Very Short Answer Questions

4. 2

6. 6

7. 3

Derivatives

UNIT 2

2.1 Introduction

In first chapter, we studied the concepts of limits and continuity. These concepts are the basic tools for differentiation. The process of finding the derivative is called differentiation. A formula of finding the derivative of a function was initiated independently by Sir Isaac Newton (1642-1727) and Gottfried W. Leibnitz (1646-1716). The differential calculus deals with the problem of finding instantaneous rate of change of a function. To determine the slope of a tangent line at an arbitrary point on a curve is the origin of derivative in geometrical sense. The derivative has great importance in calculus.

Notation

There are many ways to denote the derivative of a function $y = f(x)$. The most common notations are

$$f'(x), y', \frac{dy}{dx}, \frac{df}{dx}, \frac{d}{dx}f(x), D_x f, y'$$

- NOTE**
- Read y' as 'y prime'.
 - Read $\frac{dy}{dx}$ as 'the derivative of y with respect to x '.

The prime notations come from notations that Newton used for derivatives. The $\frac{d}{dx}$ notations are similar to those used by Leibnitz.

Input:

Function $y = f(x)$

Operator $\frac{d}{dx}$

Output:

Derivative $y' = \frac{dy}{dx}$

Increments

When the independent variable changes from one value to another, then the difference obtained by subtracting the initial value from the new value is called increment. It is also simply known as difference. An increment in x is usually denoted by Δx and is read as "delta x ". **It should be noted that Δx is a symbol representing the increment not the product of Δ and x .**

Let $y = f(x)$ be a given continuous function. Then the value of y depends upon the value of x . Hence x is an **independent variable** and y is a **dependent variable**. Let Δy be an increment in y corresponding to an increment Δx in x then,

$$y + \Delta y = f(x + \Delta x). \text{ Then, } \Delta y = f(x + \Delta x) - f(x).$$

Incremental Ratio

If y changes to $y + \Delta y$ corresponding to change of x to $x + \Delta x$, then $\frac{\Delta y}{\Delta x}$ is called incremental ratio or quotient of differences.

Definition of Derivative

Let $y = f(x)$ be a function. If Δy be an increment in y corresponding to an increment Δx in x , then $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$, if it exists, is called derivative of y with respect to x and is denoted by $\frac{dy}{dx}$. It may be denoted by $\frac{df(x)}{dx}$ or $f'(x)$.

The process of finding the differential coefficient or derivative from this method is known as the method of differentiation from the **definition** or the **first principle**.

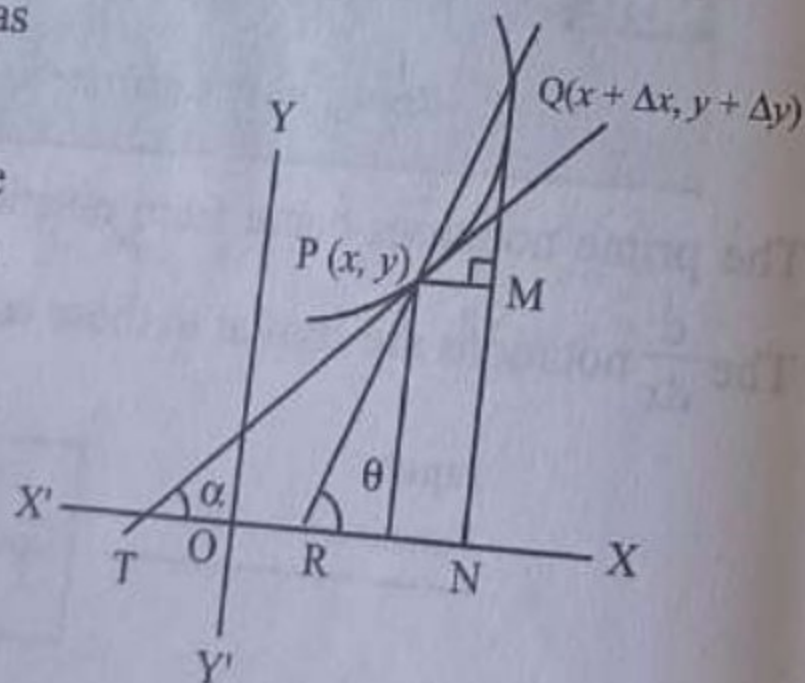
2.2 Geometrical Interpretation of Derivative

Let $y = f(x)$ be a continuous function. Let it represent a curve as shown in figure.

Let $P(x, y)$ be a point on this curve and let $Q(x + \Delta x, y + \Delta y)$ be a neighbouring point on it.

Now, slope of chord PQ

$$= \frac{(y + \Delta y) - y}{(x + \Delta x) - x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$



Let the point Q move along the curve such that $Q \rightarrow P$. As Q approaches P , the chord PQ approaches the tangent at P . This happens when $\Delta x \rightarrow 0$.

Hence,

$$\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{Q \rightarrow P} \text{(Slope of Chord } PQ)$$

or, $f'(x)$ = The slope of the tangent at P .

Thus, $\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ provided that if the limit exists.

$$\text{Hence, } \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x).$$

The process finding the differential coefficient of a function is called differentiation.

2.3 Relationship between Continuity and Differentiability

The relation between continuity and differentiability is stated in the following theorem.

Theorem: If a function f is differentiable at a point a then it is continuous at the point but not conversely.

Proof

Since f is differentiable at a , we have,

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

Now,

$$\begin{aligned} \lim_{x \rightarrow a} [f(x) - f(a)] &= \lim_{x \rightarrow a} \left[\frac{f(x) - f(a)}{x - a} \times (x - a) \right] \\ &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \times \lim_{x \rightarrow a} (x - a) \\ &= f'(a) \times 0 \\ &= 0. \end{aligned}$$

$$\text{or, } \lim_{x \rightarrow a} f(x) - f(a) = 0$$

$$\therefore \lim_{x \rightarrow a} f(x) = f(a).$$

Hence $f(x)$ is continuous at $x = a$.

The converse of the above theorem is not necessarily true.

For, let us consider the function $f(x) = |x|$.

$$\text{Here, } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} |x| = |0| = 0$$

and

$$f(0) = |0| = 0$$

Hence, $f(x) = |x|$ is continuous at $x = 0$.

But,

$$\begin{aligned} Rf'(0) &= \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{|h| - 0}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{h}{h} = 1 \end{aligned}$$

and

$$\begin{aligned} Lf'(0) &= \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{|h|}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1. \end{aligned}$$

$$\therefore Rf'(0) \neq Lf'(0).$$

Hence $f'(x)$ does not exist at $x = 0$. Thus $f(x)$ is not differentiable at $x = 0$.

2.4 Derivative of Algebraic Functions

Differentiation of x^n by Definition [Power Rule]

Let

$$y = x^n \quad \dots (i)$$

Let Δx be an increment in x and Δy be the corresponding increment in y .

Then,

$$y + \Delta y = (x + \Delta x)^n \quad \dots (ii)$$

Subtracting (i) from (ii), we get,

$$\Delta y = (x + \Delta x)^n - x^n$$

Dividing both sides by Δx , we get,

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

Taking limit when $\Delta x \rightarrow 0$, we get,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \lim_{x + \Delta x \rightarrow x} \frac{(x + \Delta x)^n - x^n}{(x + \Delta x) - x} \\ &= nx^{n-1} \quad \left[\because \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right] \end{aligned}$$

$$\text{Thus, } \frac{d}{dx} (x^n) = nx^{n-1}.$$

This is called **power rule** of derivative.

Thus,

$f(x)$	x	x^2	x^3	x^4	$\dots$	x^n	$\dots$
$f'(x)$	1	$2x$	$3x^2$	$4x^3$	$\dots$	nx^{n-1}	$\dots$

Example 1. Differentiate the following by first principle, $y = 2x^2 + 3x + 1$.

Solution

$$y = 2x^2 + 3x + 1$$

... (i)

Let Δx be an increment in x and Δy be the corresponding increment in y .

Then,

$$y + \Delta y = 2(x + \Delta x)^2 + 3(x + \Delta x) + 1 \quad \dots (ii)$$

Subtracting (i) from (ii), we get,

$$\Delta y = 2(x + \Delta x)^2 + 3(x + \Delta x) + 1 - 2x^2 - 3x - 1$$

$$\text{or, } \Delta y = 2x^2 + 4x \cdot (\Delta x) + 2(\Delta x)^2 + 3x + 3(\Delta x) + 1 - 2x^2 - 3x - 1$$

$$\text{or, } \Delta y = 2(\Delta x)^2 + 4x(\Delta x) + 3(\Delta x)$$

Dividing both sides by (Δx) , we get,

$$\frac{\Delta y}{\Delta x} = 2\Delta x + 4x + 3$$

Taking limits when $\Delta x \rightarrow 0$, we get,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2\Delta x + 4x + 3)$$

$$\therefore \frac{dy}{dx} = 4x + 3.$$

2.5 Derivative of Trigonometric Functions

1. Derivative of $\sin x$ Let $y = \sin x$ Let Δx be a small increment in x and Δy be the corresponding increment in y . Then,

$$y + \Delta y = \sin(x + \Delta x)$$

$$\text{or, } \Delta y = \sin(x + \Delta x) - y$$

$$\text{or, } \Delta y = \sin(x + \Delta x) - \sin x$$

$$\text{or, } \Delta y = 2 \cos\left(\frac{x + \Delta x + x}{2}\right) \sin\left(\frac{x + \Delta x - x}{2}\right)$$

$$\text{or, } \Delta y = 2 \cos\left(x + \frac{\Delta x}{2}\right) \sin\left(\frac{\Delta x}{2}\right)$$

Dividing both sides by Δx ,

$$\frac{\Delta y}{\Delta x} = \frac{2 \cos\left(x + \frac{\Delta x}{2}\right) \sin\left(\frac{\Delta x}{2}\right)}{\Delta x}$$

Taking limit $\Delta x \rightarrow 0$ on both sides, we get,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left\{ \frac{2 \cos\left(x + \frac{\Delta x}{2}\right) \sin\left(\frac{\Delta x}{2}\right)}{\Delta x} \right\}$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left\{ 2 \cos\left(x + \frac{\Delta x}{2}\right) \cdot \frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2} \cdot 2} \right\}$$

$$= \cos(x + 0) \cdot 1$$

$$= \cos x$$

$$\therefore \frac{d}{dx}(\sin x) = \cos x.$$

Alternative method

Let $f(x) = \sin x$ Then, $f(x + h) = \sin(x + h)$

By definition of derivative, we have,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin(x + h) - \sin x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2 \cos\left(\frac{x + h + x}{2}\right) \cdot \sin\left(\frac{x + h - x}{2}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \left\{ 2 \cos\left(x + \frac{h}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2} \cdot 2} \right\}$$

$$= \cos(x + 0)$$

$$\therefore \frac{d}{dx}(\sin x) = \cos x.$$

2. Derivative of $\cos x$ Let $f(x) = \cos x$ Then, $f(x + h) = \cos(x + h)$

By definition of derivative, we have,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cos(x + h) - \cos x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-2 \sin\left(\frac{x + h + x}{2}\right) \cdot \sin\left(\frac{x + h - x}{2}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \left\{ -2 \sin\left(x + \frac{h}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2} \cdot 2} \right\}$$

$$= -\sin(x + 0) \cdot 1$$

$$= -\sin x$$

$$\therefore \frac{d}{dx}(\cos x) = -\sin x.$$

3. Derivative of $\tan x$ Let $f(x) = \tan x$ Then, $f(x + h) = \tan(x + h)$

By definition of derivative, we have,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\tan(x+h) - \tan x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x+h) \cos x - \cos(x+h) \sin x}{h \cos(x+h) \cos x} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x+h-x)}{h \cos(x+h) \cos x} \\
 &= \lim_{h \rightarrow 0} \frac{\sin h}{h} \cdot \frac{1}{\cos(x+h) \cos x} \\
 &= 1 \cdot \frac{1}{\cos x \cdot \cos x} \\
 &= \frac{1}{\cos^2 x} = \sec^2 x
 \end{aligned}$$

$$\therefore \frac{d}{dx} (\tan x) = \sec^2 x.$$

4. Derivative of $\cot x$

Let $f(x) = \cot x$

Then, $f(x+h) = \cot(x+h)$

By definition of derivative, we have,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cot(x+h) - \cot x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin x \cos(x+h) - \cos x \sin(x+h)}{h \sin(x+h) \sin x} \\
 &= \lim_{h \rightarrow 0} \frac{\sin \{x - (x+h)\}}{h \sin(x+h) \sin x} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x-x-h)}{h \sin(x+h) \sin x}
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{\sin(-h)}{h \sin(x+h) \sin x} \\
 &= \lim_{h \rightarrow 0} \frac{-\sin h}{h \sin(x+h) \sin x} \\
 &= \lim_{h \rightarrow 0} \left\{ \frac{-\sin h}{h} \cdot \frac{1}{\sin(x+h) \sin x} \right\} \\
 &= -1 \cdot \frac{1}{\sin x \cdot \sin x} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x.
 \end{aligned}$$

$$\therefore \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x.$$

5. Derivative of $\sec x$

Let $f(x) = \sec x$

Then, $f(x+h) = \sec(x+h)$

By definition of derivative, we have,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sec(x+h) - \sec x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{\cos(x+h)} - \frac{1}{\cos x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cos x - \cos(x+h)}{h \cos x \cos(x+h)} \\
 &= \lim_{h \rightarrow 0} \frac{2 \sin \left(\frac{x+x+h}{2} \right) \cdot \sin \left(\frac{x+h-x}{2} \right)}{h \cdot \cos x \cdot \cos(x+h)} \\
 &= \lim_{h \rightarrow 0} \left\{ 2 \sin \left(x + \frac{h}{2} \right) \cdot \frac{\sin \frac{h}{2}}{\frac{h}{2} \cdot 2} \cdot \frac{1}{\cos x \cdot \cos(x+h)} \right\} \\
 &= \sin(x+0) \cdot 1 \cdot \frac{1}{\cos x \cdot \cos(x+0)} \\
 &= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \\
 &= \tan x \cdot \sec x.
 \end{aligned}$$

$$\therefore \frac{d}{dx} (\sec x) = \tan x \cdot \sec x.$$

6. Derivative of cosec x Let $f(x) = \text{cosec } x$ Then, $f(x+h) = \text{cosec}(x+h)$

By definition of derivative, we have,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\text{cosec}(x+h) - \text{cosec } x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{\sin(x+h)} - \frac{1}{\sin x}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin x - \sin(x+h)}{h \cdot \sin x \cdot \sin(x+h)}$$

$$= \lim_{h \rightarrow 0} \frac{2 \cos \left(\frac{x+x+h}{2} \right) \cdot \sin \left\{ \frac{x-(x+h)}{2} \right\}}{h \cdot \sin x \cdot \sin(x+h)}$$

$$= \lim_{h \rightarrow 0} \frac{2 \cos \left(x + \frac{h}{2} \right) \cdot \sin \left(\frac{-h}{2} \right)}{h \cdot \sin x \cdot \sin(x+h)}$$

$$= \lim_{h \rightarrow 0} \frac{-2 \cos \left(x + \frac{h}{2} \right) \cdot \sin \frac{h}{2}}{h \cdot \sin x \cdot \sin(x+h)}$$

$$= \lim_{h \rightarrow 0} \left\{ -2 \cos \left(x + \frac{h}{2} \right) \cdot \frac{\sin \frac{h}{2}}{\frac{h}{2}} \cdot \frac{1}{2 \sin x \cdot \sin(x+h)} \right\}$$

$$= -\cos x \cdot 1 \cdot \frac{1}{\sin x \cdot \sin x}$$

$$= -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$$= -\cot x \cdot \text{cosec } x$$

$$\therefore \frac{d}{dx} (\text{cosec } x) = -\cot x \cdot \text{cosec } x.$$

Summary

1. $\frac{d}{dx} (\sin x) = \cos x$

2. $\frac{d}{dx} (\cos x) = -\sin x$

3. $\frac{d}{dx} (\tan x) = \sec^2 x$

4. $\frac{d}{dx} (\cot x) = -\text{cosec}^2 x$

5. $\frac{d}{dx} (\sec x) = \sec x \tan x$

6. $\frac{d}{dx} (\text{cosec } x) = -\text{cosec } x \cot x.$

NOTE The derivative of the trigonometric functions that start with 'c' are negative.

2.6 Derivative of Logarithmic and Exponential Functions

1. Derivative of $\ln x$ Let $y = \ln x$ Let Δx and Δy be the small increments in x and y respectively. Then,

$$y + \Delta y = \ln(x + \Delta x).$$

or, $\Delta y = \ln(x + \Delta x) - y$

or, $\Delta y = \ln(x + \Delta x) - \ln x$

$$= \ln \left(\frac{x + \Delta x}{x} \right)$$

Dividing both sides by Δx , we get,

$$\frac{\Delta y}{\Delta x} = \frac{\ln \left(1 + \frac{\Delta x}{x} \right)}{\Delta x}$$

Taking limit $\Delta x \rightarrow 0$ on both sides,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\ln \left(1 + \frac{\Delta x}{x} \right)}{\Delta x}$$

$$\text{or, } \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\ln\left(1 + \frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}}$$

$$= 1 \cdot \frac{1}{x} \quad \left[\because \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1 \right]$$

$$\therefore \frac{d}{dx}(\ln x) = \frac{1}{x}$$

2. Derivative of e^x

$$\text{Let } y = e^x$$

Let Δx and Δy be the small increments in x and y respectively. Then,

$$y + \Delta y = e^{x+\Delta x}$$

$$\text{or, } \Delta y = e^{x+\Delta x} - y$$

$$\text{or, } \Delta y = e^{x+\Delta x} - e^x$$

$$\text{or, } \Delta y = e^x \cdot e^{\Delta x} - e^x$$

$$\text{or, } \Delta y = e^x (e^{\Delta x} - 1)$$

Dividing both sides by Δx ,

$$\frac{\Delta y}{\Delta x} = \frac{e^x (e^{\Delta x} - 1)}{\Delta x}$$

Taking limit $\Delta x \rightarrow 0$ on both sides, we get,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{e^x (e^{\Delta x} - 1)}{\Delta x}$$

$$\text{or, } \frac{dy}{dx} = e^x \lim_{\Delta x \rightarrow 0} \frac{e^{\Delta x} - 1}{\Delta x}$$

$$= e^x \cdot 1 \quad \left[\because \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \right]$$

$$= e^x$$

$$\therefore \frac{d}{dx}(e^x) = e^x$$

3. Derivative of a^x

$$\text{Let } y = a^x$$

Let Δx and Δy be the small increments in x and y respectively. Then,

$$y + \Delta y = a^{x+\Delta x}$$

$$\text{or, } \Delta y = a^{x+\Delta x} - y$$

$$\text{or, } \Delta y = a^{x+\Delta x} - a^x$$

$$\text{or, } \Delta y = a^x \cdot a^{\Delta x} - a^x$$

$$\text{or, } \Delta y = a^x (a^{\Delta x} - 1)$$

Dividing both sides by Δx ,

$$\frac{\Delta y}{\Delta x} = \frac{a^x (a^{\Delta x} - 1)}{\Delta x}$$

Taking limit $\Delta x \rightarrow 0$ on both sides, we get,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{a^x (a^{\Delta x} - 1)}{\Delta x}$$

$$\text{or, } \frac{dy}{dx} = a^x \lim_{\Delta x \rightarrow 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

$$= a^x \cdot \ln a \quad \left[\because \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a \right]$$

$$\therefore \frac{d}{dx}(a^x) = a^x \ln a$$

Summary

$$1. \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$2. \frac{d}{dx}(e^x) = e^x$$

$$3. \frac{d}{dx}(a^x) = a^x \ln a$$

Illustrative Examples

Example 1. Differentiate the following by first principle (definition).

a. $4x + 5$

c. $\frac{1}{x+3}$

e. $\frac{3x+5}{\sqrt{x}}$

b. $3x^2 - 2x + 5$

d. $\frac{1}{\sqrt{x}}$

Solution

a. Let $y = 4x + 5$

... (i)

Let Δx be an increment in x and Δy the corresponding increment in y . Then,

$$y + \Delta y = 4(x + \Delta x) + 5$$

$$y + \Delta y = 4x + 4\Delta x + 5$$

... (ii)

Subtracting (i) from (ii) we get, $\Delta y = 4 \Delta x$

or $\frac{\Delta y}{\Delta x} = 4$

Taking limit when $\Delta x \rightarrow 0$, we get,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} 4 \quad (4)$$

$$\therefore \frac{dy}{dx} = 4.$$

b. Let $y = 3x^2 - 2x + 5$... (i)

Let Δx be an increment in x and Δy the corresponding increment in y . Then

$$y + \Delta y = 3(x + \Delta x)^2 - 2(x + \Delta x) + 5$$

$$y + \Delta y = 3x^2 + 6x(\Delta x) + 3(\Delta x)^2 - 2x - 2(\Delta x) + 5 \dots (ii)$$

Subtracting (i) from (ii),

$$\Delta y = 6x(\Delta x) + 3(\Delta x)^2 - 2(\Delta x)$$

Dividing both sides by Δx ,

$$\frac{\Delta y}{\Delta x} = 6x + 3(\Delta x) - 2$$

Taking limit when $\Delta x \rightarrow 0$,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} (6x + 3\Delta x - 2)$$

$$\therefore \frac{dy}{dx} = 6x - 2.$$

c. Let $y = \frac{1}{x+3}$

Also, let Δx and Δy be the small increments in x and y respectively. Then

$$y + \Delta y = \frac{1}{x + \Delta x + 3}$$

or, $\Delta y = \frac{1}{x + \Delta x + 3} - y$

or, $\Delta y = \frac{1}{x + \Delta x + 3} - \frac{1}{x + 3}$

or, $\Delta y = \frac{x + 3 - x - \Delta x - 3}{(x + \Delta x + 3)(x + 3)}$

or, $\Delta y = \frac{-\Delta x}{(x + \Delta x + 3)(x + 3)}$

Dividing both sides by Δx ,

$$\frac{\Delta y}{\Delta x} = \frac{-\Delta x}{\Delta x(x + \Delta x + 3)(x + 3)}$$

$$= \frac{-1}{(x + \Delta x + 3)(x + 3)}$$

By definition of derivative,

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-1}{(x + \Delta x + 3)(x + 3)} \\ &= \frac{-1}{(x + 0 + 3)(x + 3)} \\ &= -\frac{1}{(x + 3)^2}. \end{aligned}$$

d. Let $y = \frac{1}{\sqrt{x}}$

Also, let Δx and Δy be the small increments of x and y respectively. Then

$$y + \Delta y = \frac{1}{\sqrt{x + \Delta x}}$$

or, $\Delta y = \frac{1}{\sqrt{x + \Delta x}} - \frac{1}{\sqrt{x}}$

or, $\Delta y = \frac{\sqrt{x} - \sqrt{x + \Delta x}}{\sqrt{x}(\sqrt{x + \Delta x})} \times \frac{\sqrt{x} + \sqrt{x + \Delta x}}{\sqrt{x} + \sqrt{x + \Delta x}}$

or, $\Delta y = \frac{x - x - \Delta x}{\sqrt{x}(\sqrt{x + \Delta x})(\sqrt{x} + \sqrt{x + \Delta x})}$

or, $\frac{\Delta y}{\Delta x} = \frac{-\Delta x}{\Delta x(\sqrt{x}(\sqrt{x + \Delta x})(\sqrt{x} + \sqrt{x + \Delta x}))}$

or, $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-1}{\sqrt{x}(\sqrt{x + \Delta x})(\sqrt{x} + \sqrt{x + \Delta x})}$

$$\therefore \frac{dy}{dx} = \frac{-1}{\sqrt{x}(\sqrt{x + 0})(\sqrt{x} + \sqrt{x + 0})}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{x(2\sqrt{x})} = -\frac{1}{2x^{3/2}}.$$

e. Let $y = \frac{3x+5}{\sqrt{x}} = \frac{3x}{\sqrt{x}} + \frac{5}{\sqrt{x}} = 3\sqrt{x} + \frac{5}{\sqrt{x}}$

Also, let Δx and Δy be the small increments in x and y respectively. Then,

$$y + \Delta y = 3\sqrt{x + \Delta x} + \frac{5}{\sqrt{x + \Delta x}}$$

or, $\Delta y = 3\sqrt{x + \Delta x} + \frac{5}{\sqrt{x + \Delta x}} - y$

or, $\Delta y = 3\sqrt{x + \Delta x} + \frac{5}{\sqrt{x + \Delta x}} - 3\sqrt{x} - \frac{5}{\sqrt{x}}$

or, $\Delta y = 3(\sqrt{x + \Delta x} - \sqrt{x}) + 5\left(\frac{1}{\sqrt{x + \Delta x}} - \frac{1}{\sqrt{x}}\right)$

$$\text{or, } \Delta y = 3 \frac{(\sqrt{x+\Delta x} - \sqrt{x})}{1} \times \frac{(\sqrt{x+\Delta x} + \sqrt{x})}{(\sqrt{x+\Delta x} + \sqrt{x})} + 5 \left(\frac{\sqrt{x} - \sqrt{x+\Delta x}}{\sqrt{x}\sqrt{x+\Delta x}} \right)$$

$$\text{or, } \Delta y = \frac{3(x+\Delta x-x)}{\sqrt{x+\Delta x} + \sqrt{x}} + \frac{5(\sqrt{x} - \sqrt{x+\Delta x})}{\sqrt{x}\sqrt{x+\Delta x}} \times \frac{(\sqrt{x} + \sqrt{x+\Delta x})}{(\sqrt{x} + \sqrt{x+\Delta x})}$$

$$\text{or, } \Delta y = \frac{3\Delta x}{\sqrt{x+\Delta x} + \sqrt{x}} + \frac{5(x-x-\Delta x)}{\sqrt{x}\sqrt{x+\Delta x}(\sqrt{x} + \sqrt{x+\Delta x})}$$

$$\text{or, } \Delta y = \frac{3\Delta x}{\sqrt{x+\Delta x} + \sqrt{x}} - \frac{5\Delta x}{\sqrt{x}\sqrt{x+\Delta x}(\sqrt{x} + \sqrt{x+\Delta x})}$$

$$\text{or, } \Delta y = \Delta x \left\{ \frac{3}{\sqrt{x+\Delta x} + \sqrt{x}} - \frac{5}{\sqrt{x}\sqrt{x+\Delta x}(\sqrt{x} + \sqrt{x+\Delta x})} \right\}$$

$$\text{or, } \frac{\Delta y}{\Delta x} = \frac{3}{\sqrt{x+\Delta x} + \sqrt{x}} - \frac{5}{\sqrt{x}\sqrt{x+\Delta x}(\sqrt{x} + \sqrt{x+\Delta x})}$$

Taking limit $\Delta x \rightarrow 0$ on both sides, we get

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left\{ \frac{3}{\sqrt{x+\Delta x} + \sqrt{x}} - \frac{5}{\sqrt{x}\sqrt{x+\Delta x}(\sqrt{x} + \sqrt{x+\Delta x})} \right\}$$

$$\frac{dy}{dx} = \frac{3}{\sqrt{x} + \sqrt{x}} - \frac{5}{\sqrt{x}\sqrt{x}(\sqrt{x} + \sqrt{x})}$$

$$= \frac{3}{2\sqrt{x}} - \frac{5}{2x\sqrt{x}}$$

$$= \frac{3x-5}{2x\sqrt{x}}$$

$$= \frac{3x-5}{2x^{3/2}}$$

Example 2. Find from definition, the derivatives of $\sin 2x$.

Solution

Let, $y = \sin 2x$

Let, Δx and Δy be the small increments in x and y respectively. Then

$$y + \Delta y = \sin 2(x + \Delta x) = \sin (2x + 2\Delta x)$$

$$\text{or, } \Delta y = \sin 2(x + \Delta x) - y$$

$$\text{or, } \Delta y = \sin (2x + 2\Delta x) - \sin 2x$$

$$\text{or, } \Delta y = 2 \cos \left(\frac{2x + 2\Delta x + 2x}{2} \right) \sin \left(\frac{2x + 2\Delta x - 2x}{2} \right)$$

$$= \cos (2x + \Delta x) \sin \Delta x$$

Dividing both sides by Δx , we get,

$$\frac{\Delta y}{\Delta x} = 2 \cos (2x + \Delta x) \left(\frac{\sin \Delta x}{\Delta x} \right)$$

Now, from the definition, we have,

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right)$$

$$= \lim_{\Delta x \rightarrow 0} 2 \cos (2x + \Delta x) \left(\frac{\sin \Delta x}{\Delta x} \right)$$

$$= 2 \cos (2x + 0) \times 1$$

$$= 2 \cos 2x$$

$$\therefore \frac{d}{dx} (\sin 2x) = 2 \cos 2x.$$

Example 3. Find $\frac{dy}{dx}$ from first principles of $y = \sqrt{\tan x}$

Solution

We have

$$y = \sqrt{\tan x}$$

Let Δx and Δy be the small increments in x and y respectively. Then,

$$\Delta y = \sqrt{\tan (x + \Delta x)}$$

$$\Delta y = \sqrt{\tan (x + \Delta x)} - y$$

$$\text{or, } \Delta y = \sqrt{\tan (x + \Delta x)} - \sqrt{\tan x}$$

$$= (\sqrt{\tan (x + \Delta x)} - \sqrt{\tan x}) \times \frac{(\sqrt{\tan (x + \Delta x)} + \sqrt{\tan x})}{(\sqrt{\tan (x + \Delta x)} + \sqrt{\tan x})}$$

$$= \frac{\tan (x + \Delta x) - \tan x}{\sqrt{\tan (x + \Delta x)} + \sqrt{\tan x}}$$

$$= \frac{\frac{\sin (x + \Delta x)}{\cos (x + \Delta x)} - \frac{\sin x}{\cos x}}{\sqrt{\tan (x + \Delta x)} + \sqrt{\tan x}}$$

$$= \frac{\sin (x + \Delta x) \cos x - \cos (x + \Delta x) \sin x}{\cos (x + \Delta x) \cos x (\sqrt{\tan (x + \Delta x)} + \sqrt{\tan x})}$$

$$= \frac{\sin (x + \Delta x - x)}{\cos (x + \Delta x) \cos x (\sqrt{\tan (x + \Delta x)} + \sqrt{\tan x})}$$

$$= \frac{\sin \Delta x}{\cos (x + \Delta x) \cos x (\sqrt{\tan (x + \Delta x)} + \sqrt{\tan x})}$$

$$\Delta y = \frac{\sin \Delta x}{\cos (x + \Delta x) \cos x (\sqrt{\tan (x + \Delta x)} + \sqrt{\tan x})}$$

Dividing both sides by Δx , we get,

$$\frac{\Delta y}{\Delta x} = \frac{\sin \Delta x}{\Delta x} \cdot \frac{1}{\cos (x + \Delta x) \cos x (\sqrt{\tan (x + \Delta x)} + \sqrt{\tan x})}$$

From definition of derivative, we have,

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x}{\Delta x} \times \frac{1}{\cos(x+\Delta x)\cos x(\sqrt{\tan(x+\Delta x)} + \sqrt{\tan x})} \\
 &= 1 \times \frac{1}{\cos(x+0)\cos x(\sqrt{\tan(x+0)} + \sqrt{\tan x})} \\
 &= \frac{1}{\cos x \cos x(\sqrt{\tan x} + \sqrt{\tan x})} \\
 &= \frac{1}{\cos^2 x \cdot 2\sqrt{\tan x}} = \frac{1}{\cos^2 x} \cdot \frac{1}{2\sqrt{\tan x}} \\
 \therefore \frac{dy}{dx} &= \frac{\sec^2 x}{2\sqrt{\tan x}}
 \end{aligned}$$

Example 4. Find from first principles the derivatives of e^{ax+b} .

Solution

Let, $y = e^{ax+b}$

Also, let Δx and Δy be the small increments in x and y respectively.

Then,

$$y + \Delta y = e^{a(x+\Delta x)+b}$$

$$\text{or, } \Delta y = e^{a(x+\Delta x)+b} - y$$

$$\text{or, } \Delta y = e^{ax+ab+b} - e^{ax+b}$$

$$\text{or, } \Delta y = e^{ax+b} \cdot e^{a\Delta x} - e^{ax+b}$$

$$\text{or, } \Delta y = e^{ax+b}(e^{a\Delta x} - 1)$$

Dividing both sides by Δx , we get,

$$\frac{\Delta y}{\Delta x} = \frac{e^{ax+b}(e^{a\Delta x} - 1)}{\Delta x}$$

Taking limit $\Delta x \rightarrow 0$ on both sides, we have,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{e^{ax+b}(e^{a\Delta x} - 1)}{\Delta x}$$

$$\begin{aligned}
 \therefore \frac{\Delta y}{\Delta x} &= e^{ax+b} \lim_{\Delta x \rightarrow 0} \frac{e^{a\Delta x} - 1}{a\Delta x} \cdot a \\
 &= e^{ax+b} \cdot 1 \cdot a
 \end{aligned}$$

$$\therefore \frac{d}{dx}(e^{ax+b}) = a \cdot e^{ax+b}$$

Exercise 2.1

1. Find the derivatives from first principle (or by definition).

a. $2x+5$

b. x^2+x

c. $(x-1)^2$

d. $2x^2+3x+1$

e. $x+\frac{1}{x}$

f. $\frac{ax+b}{x}$

g. $\frac{1}{x+2}$

h. $\frac{1}{2-3x}$

i. $\sqrt{x}$

j. $\sqrt{2x+3}$

k. $x+\sqrt{x}$

l. $\frac{1}{\sqrt{x}}$

m. $\sqrt{\frac{1}{1-x}}$

n. $\frac{1}{\sqrt{2x-5}}$

2. Find, from definition, the derivative of

a. $\sin 3x$

b. $\cos(3x-4)$

c. $\tan \frac{3x}{2}$

d. $\sin^2 x$

e. $\sqrt{\sin 2x}$

f. $\sqrt{\sec x}$

3. Find, from definition, the derivatives of

a. e^{ax}

b. e^{2x+3}

c. $\ln(3x+5)$

d. $\log_a x$

Very Short Answer Questions

1. Define derivative of a function at a point.
2. What is the geometrical meaning of derivative of function $f(x)$ at $x=a$?
3. Write the relation between derivative and continuity.
4. What is the slope of line joining the points (x, y) and $(x+\Delta x, y+\Delta y)$?
5. What do you mean by differentiation?

Answers

1. a. 2

b. $2x+1$

c. $2x-2$

d. $4x+3$

e. $1-\frac{1}{x^2}$

f. $-\frac{b}{x^2}$

g. $-\frac{1}{(x+2)^2}$

h. $\frac{3}{(2-3x)^2}$

i. $\frac{1}{2\sqrt{x}}$

j. $\frac{1}{\sqrt{2x+3}}$

k. $1+\frac{1}{2\sqrt{x}}$

l. $-\frac{1}{2x^{3/2}}$

m. $\frac{1}{2(1-x)^{3/2}}$

n. $-\frac{1}{(2x-5)^{3/2}}$

2. a. $3 \cos 3x$

b. $-3 \sin(3x-4)$

c. $\frac{3}{2} \sec^2 \frac{3x}{2}$

d. $\sin 2x$

e. $\frac{\cos 2x}{\sqrt{\sin 2x}}$

f. $\frac{1}{2} \tan x \sqrt{\sec x}$

3. a. ae^{ax}

b. $2e^{2x+3}$

c. $\frac{3}{3x+5}$

d. $\frac{\log_a e}{x}$

2.7 Rules of Differentiation

1. The derivative of a constant function is zero i.e. $\frac{d(c)}{dx} = 0$, where c is constant.

2. The derivative of a scalar multiple of a function is the scalar multiple of the derivative of the function.

$$\frac{d}{dx}(ku) = k \frac{d}{dx}(u), \text{ where } k \text{ is a constant and } u \text{ is a function of } x.$$

3. The derivative of the sum of two functions is the sum of their derivatives (Sum rule).

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}, \text{ where } u \text{ and } v \text{ are functions of } x.$$

This rule may also be extended further as

$$\frac{d}{dx}(u+v+w) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}, \text{ where } u, v \text{ and } w \text{ are functions of } x.$$

4. Derivative of the product of functions (Product Rule)

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}, \text{ where } u \text{ and } v \text{ are functions of } x.$$

It may be remembered as;

Derivative of the product of two functions

$$= (1^{\text{st}} \text{ function}) \times (\text{Derivative of } 2^{\text{nd}} \text{ function}) + (2^{\text{nd}} \text{ function}) \times (\text{Derivative of } 1^{\text{st}} \text{ function})$$

5. Derivative of Quotient of two functions (Quotient Rule)

Let, $y = \frac{u}{v}$, both u and v are functions of x .

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

It may be remembered as

Derivative of the quotient of two functions

$$= \frac{(\text{Denominator} \times \text{Derivative of numerator}) - (\text{Numerator} \times \text{Derivative of denominator})}{(\text{Denominator})^2}$$

6. Derivative of a function of a function (Chain rule)

If y is a function of t , $y = f(t)$ and again t is a function of x , $t = g(x)$, then the derivative of y with respect to x is given by

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

This result may be extended further as

It $y = f(t)$, $t = g(u)$ and $u = h(x)$

$$\text{Then } \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}$$

Example 1. Find $\frac{dy}{dx}$, if $y = z^3 + 2z + 1$, $x = z^2 + 2$.

Solution

Here

$$y = z^3 + 2z + 1$$

$$\therefore \frac{dy}{dz} = \frac{d}{dz}(z^3 + 2z + 1) \\ = 3z^2 + 2$$

Again,

$$x = z^2 + 2$$

Now,

$$\frac{dx}{dz} = 2z$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = (3z^2 + 2) \times \frac{1}{2z} \\ = \frac{3z^2 + 2}{2z}$$

2.8 Differentiation of Parametric Functions

Sometimes x and y are given as functions of another variable t , then ' t ' is called parameter i.e. $x = f(t)$ and $y = g(t)$.

The derivative of y with respect to x is

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}, \frac{dx}{dt} \neq 0.$$

Example 2. Find $\frac{dy}{dx}$ when $x = a \frac{1-t^2}{1+t^2}$, $y = b \frac{2t}{1+t^2}$.

Solution

We have

$$x = a \frac{1-t^2}{1+t^2}$$

$$\therefore \frac{dx}{dt} = a \frac{(1+t^2)(-2t) - (1-t^2)2t}{(1+t^2)^2} = -\frac{4at}{(1+t^2)^2}$$

$$\text{Again, } y = b \frac{2t}{1+t^2}$$

$$\therefore \frac{dy}{dt} = b \frac{(1+t^2)2 - 2t \times t}{(1+t^2)^2} = \frac{2b(1-t^2)}{(1+t^2)^2}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{b}{2at} (1-t^2).$$

2.9 Differentiation of Implicit Functions

If both variables x and y are expressed in a functional relation, then either variable is an implicit function of the other.

If $x^2 + y^2 + 3axy + 3bx = 0$; then, x is an implicit function of y and vice-versa.

In this case, we differentiate each term of the given function with respect to x and find $\frac{dy}{dx}$.

Example 3. Find $\frac{dy}{dx}$ if $x^2 + y^2 + 3axy + 3bx = 0$.

Solution

Differentiating w.r. to x on both sides, we get,

$$\frac{d}{dx} (x^2 + y^2 + 3axy + 3bx) = \frac{d}{dx} (0)$$

$$\text{or, } \frac{dx^2}{dx} + \frac{dy^2}{dy} \cdot \frac{dy}{dx} + 3a \times \frac{d(xy)}{dx} + 3b \frac{dx}{dx} = 0$$

$$\text{or, } 2x + 2y \frac{dy}{dx} + 3a \left(x \frac{dy}{dx} + y \frac{dx}{dx} \right) + 3b = 0$$

$$\text{or, } 2x + 2y \frac{dy}{dx} + 3ax \frac{dy}{dx} + 3ay + 3b = 0$$

$$\text{or, } \frac{dy}{dx} (3ax + 2y) + (2x + 3ay + 3b) = 0$$

$$\therefore \frac{dy}{dx} = -\frac{2x + 3ay + 3b}{3ax + 2y}$$

LIST OF IMPORTANT FORMULAE

1. $\frac{d}{dx} (x^n) = nx^{n-1}$
2. $\frac{d}{dx} (\text{constant}) = 0$
3. $\frac{d}{dx} (u \pm v) = \frac{d}{dx} (u) \pm \frac{d}{dx} (v)$
4. $\frac{d}{dx} (uv) = u \frac{d}{dx} (v) + v \frac{d}{dx} (u)$
5. $\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{d}{dx} (u) - u \frac{d}{dx} (v)}{v^2}$

2.10 Higher Order Derivatives

Let $y = f(x)$ be a differentiable function. Then its derivative denoted by $\frac{dy}{dx}$ or $f'(x)$ is called the first derivative of $f(x)$ with respect to x . The derivative of $\frac{dy}{dx}$ or $f'(x)$ is called the second derivative of $y = f(x)$ with respect to x and is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$. The third derivative of $y = f(x)$ is denoted by $\frac{d^3y}{dx^3}$ or $f'''(x)$, and so on for derivatives of any order. For the n^{th} derivative, we denote it by $f^{(n)}(x)$ or $\frac{d^n y}{dx^n}$.

These higher derivatives are also called **successive derivatives**.

Example 4. If $y = x^3 - 2x^2 + 3x + 5$, then find the successive derivatives of y .

Solution

We have,

$$y = x^3 - 2x^2 + 3x + 5 \quad \dots (i)$$

Differentiating both sides w.r.t. 'x', we have,

$$\frac{dy}{dx} = \frac{d}{dx} (x^3 - 2x^2 + 3x + 5)$$

$$= 3x^2 - 4x + 3$$

$$\therefore \frac{dy}{dx} = 3x^2 - 4x + 3 \quad \dots (ii)$$

Differentiating (ii), we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$= \frac{d}{dx} (3x^2 - 4x + 3)$$

$$= 6x - 4$$

$$\therefore \frac{d^2y}{dx^2} = 6x - 4 \quad \dots (iii)$$

Differentiating (iii), we get,

$$\frac{d^3y}{dx^3} = \frac{d}{dx} \left(\frac{d^2y}{dx^2} \right)$$

$$= \frac{d}{dx} (6x - 4)$$

$$= 6$$

$$\therefore \frac{d^3y}{dx^3} = 6 \quad \dots (iv)$$

Differentiating (iv), we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$= \frac{d(6)}{dx}$$

$$= 0$$

$$\therefore \frac{d^2y}{dx^2} = 0.$$

Illustrative Examples

Example 1. Find the differential coefficient of

a. $3x^3 + 2x^2 + x + 5$

b. $x + \frac{1}{x}$

c. $\frac{x^8}{8} + \frac{x^6}{6} - \frac{x^4}{4} + 2$

Solution

a. Let $y = 3x^3 + 2x^2 + x + 5$

Differentiating both sides with respect to x , we get,

$$\frac{dy}{dx} = \frac{d}{dx} (3x^3 + 2x^2 + x + 5)$$

$$= 3 \frac{dx^3}{dx} + 2 \frac{dx^2}{dx} + \frac{dx}{dx} + \frac{d}{dx} (5)$$

$$= 9x^2 + 4x + 1 + 0$$

$$= 9x^2 + 4x + 1.$$

b. Let $y = x + \frac{1}{x}$

Differentiating both sides w.r. to x ,

$$\frac{dy}{dx} = \frac{d}{dx} \left(x + \frac{1}{x} \right)$$

$$= \frac{dx}{dx} + \frac{d}{dx} \left(\frac{1}{x} \right)$$

$$= 1 + \frac{dx^{-1}}{dx}$$

$$= 1 - \frac{1}{x^2}.$$

c. Let $y = \frac{x^8}{8} + \frac{x^6}{6} - \frac{x^4}{4} + 2$

Differentiating both sides w.r. to ' x ', we get,

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^8}{8} + \frac{x^6}{6} - \frac{x^4}{4} + 2 \right)$$

$$= \frac{1}{8} \frac{dx^8}{dx} + \frac{1}{6} \frac{dx^6}{dx} - \frac{1}{4} \frac{dx^4}{dx} + \frac{d(2)}{dx}$$

$$= \frac{1}{8} \cdot 8x^7 + \frac{1}{6} \cdot 6x^5 - \frac{1}{4} \cdot 4x^3 + 0$$

$$= x^7 + x^5 - x^3.$$

Example 2. Find $\frac{dy}{dx}$ if:

a. $y = x^2(x^3 + 1)$

b. $y = x^2\sqrt{1+x^2}$

Solution

a. Let $y = x^2(x^3 + 1)$

Differentiating both sides w.r. to ' x '

$$\frac{dy}{dx} = \frac{d}{dx} \{x^2(x^3 + 1)\}$$

$$= x^2 \frac{d(x^3 + 1)}{dx} + (x^3 + 1) \frac{dx^2}{dx} \quad \left(\because \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \right)$$

$$= x^2 \cdot 3x^2 + (x^3 + 1) \cdot 2x$$

$$= 3x^4 + 2x^4 + 2x$$

$$= 5x^4 + 2x.$$

b. $y = x^2\sqrt{1+x^2}$

Differentiating both sides w.r. to ' x ', we get,

$$\frac{dy}{dx} = \frac{d}{dx} (x^2\sqrt{1+x^2})$$

$$= x^2 \frac{d(1+x^2)^{1/2}}{dx} + \sqrt{1+x^2} \frac{dx^2}{dx} \quad \left(\because \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \right)$$

$$= x^2 \cdot \frac{d(1+x^2)^{1/2}}{d(1+x^2)} \cdot \frac{d(1+x^2)}{dx} + 2x\sqrt{1+x^2}$$

$$= x^2 \cdot \frac{1}{2} (1+x^2)^{-1/2} \cdot 2x + 2x\sqrt{1+x^2}$$

$$= \frac{x^3}{\sqrt{1+x^2}} + 2x\sqrt{1+x^2}$$

$$= \frac{x^3 + 2x + 2x^3}{\sqrt{1+x^2}}$$

$$= \frac{3x^3 + 2x}{\sqrt{1+x^2}}.$$

Example 3. Find the derivative of

a. $y = \frac{x^3+1}{x^2-1}$

b. $y = \frac{x^2}{x-1}$

Solution

a. $y = \frac{x^3+1}{x^2-1}$

Differentiating both sides w.r. to 'x', we get,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(\frac{x^3+1}{x^2-1} \right) \\ &= \frac{(x^2-1) \frac{d}{dx}(x^3+1) - (x^3+1) \frac{d}{dx}(x^2-1)}{(x^2-1)^2} \left(\because \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right) \\ &= \frac{(x^2-1) 3x^2 - (x^3+1) 2x}{(x^2-1)^2} \\ &= \frac{3x^4 - 3x^2 - 2x^4 - 2x}{(x^2-1)^2} \\ &= \frac{x^4 - 3x^2 - 2x}{(x^2-1)^2}\end{aligned}$$

b. $y = \frac{x^2}{x-1}$

Differentiating both sides w.r. to 'x', we get,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(\frac{x^2}{x-1} \right) \\ &= \frac{(x-1) \frac{d}{dx} x^2 - x^2 \frac{d}{dx} (x-1)}{(x-1)^2} \\ &= \frac{(x-1) 2x - x^2 \times 1}{(x-1)^2} \\ &= \frac{2x^2 - 2x - x^2}{(x-1)^2} \\ &= \frac{x^2 - 2x}{(x-1)^2}\end{aligned}$$

Example 4. Find the derivative of

a. $y = \frac{1}{(x^2-2x+1)^{1/3}}$

b. $y = \sqrt{ax^2+bx+c}$

Solution

a. $y = \frac{1}{(x^2-2x+1)^{1/3}}$
 $= (x^2-2x+1)^{-1/3}$

Differentiating both sides w.r. to 'x'

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^2-2x+1)^{-1/3}}{dx} \\ &= \frac{d(x^2-2x+1)^{-1/3}}{d(x^2-2x+1)} \cdot \frac{d(x^2-2x+1)}{dx} \\ &= -\frac{1}{3} (x^2-2x+1)^{-1/3-1} (2x-2) \\ &= -\frac{(2x-2)}{3(x^2-2x+1)^{4/3}}\end{aligned}$$

b. $y = \sqrt{ax^2+bx+c}$

Differentiating both sides w.r. to 'x'

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(ax^2+bx+c)^{1/2}}{dx} \\ &= \frac{d(ax^2+bx+c)^{1/2}}{d(ax^2+bx+c)} \cdot \frac{d(ax^2+bx+c)}{dx} \\ &= \frac{1}{2} (ax^2+bx+c)^{1/2-1} (2ax+b) \\ &= \frac{2ax+b}{2\sqrt{ax^2+bx+c}}\end{aligned}$$

Example 5. Find the derivative of y w.r. to x in each of the following:

a. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

b. $x^5 + y^5 + 5xy - 15 = 0$

Solution

a. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Differentiating both sides w.r. to 'x' we get

$$\frac{d}{dx} (ax^2 + 2hxy + by^2 + 2gx + 2fy + c) = 0$$

$$\text{or, } 2ax + 2h \frac{d}{dx} (xy) + b \frac{dy^2}{dx} + 2g + 2f \frac{dy}{dx} + 0 = 0$$

$$\text{or, } 2ax + 2h \left(x \cdot \frac{dy}{dx} + y \cdot \frac{dx}{dx} \right) + b \frac{dy^2}{dy} \cdot \frac{dy}{dx} + 2g + 2f \frac{dy}{dx} = 0$$

$$\text{or, } 2ax + 2hx \frac{dy}{dx} + 2hy + 2by \frac{dy}{dx} + 2g + 2f \frac{dy}{dx} = 0$$

$$\text{or, } (2ax + 2hy + 2g) + \frac{dy}{dx} (2hx + 2by + 2f) = 0$$

$$\text{or, } \frac{dy}{dx} = -\frac{2ax + 2hy + 2g}{2hx + 2by + 2f}$$

$$= -\frac{ax + hy + g}{hx + by + f}$$

b. $x^5 + y^5 + 5xy - 15 = 0$
Differentiating both sides w.r.t. 'x'

$$\frac{dy}{dx} (x^5 + y^5 + 5xy - 15) = 0$$

$$\text{or, } \frac{dx^5}{dx} + \frac{dy^5}{dx} + \frac{5d(xy)}{dx} - \frac{d(15)}{dx} = 0$$

$$\text{or, } 5x^4 + \frac{dy^5}{dx} + 5 \left(x \frac{dy}{dx} + y \frac{dx}{dx} \right) - 0 = 0$$

$$\text{or, } 5x^4 + 5y^4 \cdot \frac{dy}{dx} + 5x \frac{dy}{dx} + 5y = 0$$

$$\text{or, } (5x^4 + 5y) + \frac{dy}{dx} (5y^4 + 5x) = 0$$

$$\text{or, } \frac{dy}{dx} = -\frac{5x^4 + 5y}{5y^4 + 5x}$$

$$= -\frac{x^4 + y}{y^4 + x}$$

Example 6. a. If $x = 2at$, $y = at^2$, find $\frac{dy}{dx}$

b. If $x = at^3$, $y = b\left(t - \frac{1}{t}\right)$, find $\frac{dy}{dx}$.

Solution

a. Here, $x = 2at$

Differentiating both sides w.r.t. 't'

$$\frac{dx}{dt} = 2a$$

Again, $y = at^2$

Differentiating both sides w.r.t. 't'

$$\frac{dy}{dt} = a \frac{d(t^2)}{dt} = 2at$$

Now,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2at}{2a} = t$$

b. Here, $x = at^3$

Differentiating both sides w.r.t. 't', we get,

$$\frac{dx}{dt} = a \frac{d(t^3)}{dt} = 3at^2$$

Again, $y = b\left(t - \frac{1}{t}\right)$

Differentiating both sides w.r.t. 't', we get,

$$\frac{dy}{dt} = b \left(\frac{dt}{dt} - \frac{d(t^{-1})}{dt} \right)$$

$$= b + \frac{b}{t^2}$$

$$= b \left(1 + \frac{1}{t^2} \right)$$

Now,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= b \left(1 + \frac{1}{t^2} \right) \cdot \frac{1}{3at^2}$$

$$= \frac{b(t^2 + 1)}{3at^4}$$

Example 7. a. Find the derivative of $y = \frac{1}{\sqrt{x+a} - \sqrt{x}}$

b. Find the derivative of $y = \frac{1}{\sqrt{2x-3} - \sqrt{2x-5}}$

Solution

a. $y = \frac{1}{\sqrt{x+a} - \sqrt{x}} = \frac{1}{\sqrt{x+a} - \sqrt{x}} \times \frac{\sqrt{x+a} + \sqrt{x}}{\sqrt{x+a} + \sqrt{x}}$
 $= \frac{\sqrt{x+a} + \sqrt{x}}{x+a-x} = \frac{1}{a} (\sqrt{x+a} + \sqrt{x})$

Differentiating both sides w.r.t. 'x', we get,

$$\frac{dy}{dx} = \frac{1}{a} \frac{d}{dx} (\sqrt{x+a} + \sqrt{x})$$

$$= \frac{1}{a} \left\{ \frac{d}{dx} (x+a)^{\frac{1}{2}} + \frac{dx^{\frac{1}{2}}}{dx} \right\}$$

$$= \frac{1}{a} \left\{ \frac{d}{dx} (x+1)^{\frac{1}{2}} + \frac{1}{2} x^{-\frac{1}{2}} \right\}$$

$$= \frac{1}{2a} \left(\frac{1}{\sqrt{x+a}} + \frac{1}{\sqrt{x}} \right)$$

b. $y = \frac{1}{\sqrt{2x-3} - \sqrt{2x-5}}$

$$= \frac{1}{\sqrt{2x-3} - \sqrt{2x-5}} \times \frac{\sqrt{2x-3} + \sqrt{2x-5}}{\sqrt{2x-3} + \sqrt{2x-5}}$$

$$= \frac{\sqrt{2x-3} + \sqrt{2x-5}}{2x-3-2x+5}$$

$$= \frac{1}{2} (\sqrt{2x-3} + \sqrt{2x-5})$$

Differentiating both sides w.r. to 'x', we get,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2} \frac{d}{dx} (\sqrt{2x-3} + \sqrt{2x-5}) \\ &= \frac{1}{2} \left\{ \frac{d}{dx} (2x-3)^{\frac{1}{2}} + \frac{d}{dx} (2x-5)^{\frac{1}{2}} \right\} \\ &= \frac{1}{2} \left\{ \frac{d(2x-3)}{dx} \cdot \frac{d(2x-3)}{dx} + \frac{d(2x-5)}{dx} \cdot \frac{d(2x-5)}{dx} \right\} \\ &= \frac{1}{2} \left\{ \frac{1}{2} (2x-3)^{-\frac{1}{2}} \times 2 + \frac{1}{2} (2x-5)^{-\frac{1}{2}} \times 2 \right\} \\ &= \frac{1}{2} \left(\frac{1}{\sqrt{2x-3}} + \frac{1}{\sqrt{2x-5}} \right).\end{aligned}$$

Example 8. Find $\frac{dy}{dx}$ if $y = 3x + \cot x$.

Solution

Here,

$$y = 3x + \cot x$$

Differentiating both sides w.r. to x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} (3x + \cot x) \\ &= \frac{d}{dx} (3x) + \frac{d}{dx} (\cot x) \\ &= 3 + \operatorname{cosec}^2 x.\end{aligned}$$

Example 9. Find $\frac{dy}{dx}$ when $y = \sec^2(\tan \sqrt{x})$.

Solution

Here,

$$y = \sec^2(\tan \sqrt{x})$$

Differentiating both sides with respect to 'x', we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \{\sec^2(\tan \sqrt{x})\} \\ &= \frac{d \{\sec(\tan \sqrt{x})\}^2}{d \{\sec(\tan \sqrt{x})\}} \cdot \frac{d \{\sec(\tan \sqrt{x})\}}{d(\tan \sqrt{x})} \cdot \frac{d(\tan \sqrt{x})}{d(\sqrt{x})} \cdot \frac{d(\sqrt{x})}{dx} \\ &= 2 \sec(\tan \sqrt{x}) \cdot \sec(\tan \sqrt{x}) \tan(\tan \sqrt{x}) \cdot \sec^2 \sqrt{x} \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{1}{\sqrt{x}} \sec^2(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \cdot \sec^2 \sqrt{x}.\end{aligned}$$

Example 10. Find $\frac{dy}{dx}$, when $x = a(1 - \cos t)$ and $y = a(t + \sin t)$.

Solution

Here,

$$y = a(t + \sin t)$$

$$\begin{aligned}\frac{dy}{dt} &= a \left\{ \frac{d(t)}{dt} + \frac{d(\sin t)}{dt} \right\} \\ &= a(1 + \cos t)\end{aligned}$$

And, $x = a(1 - \cos t)$

$$\begin{aligned}\frac{dx}{dt} &= a \left\{ \frac{d(1)}{dt} - \frac{d(\cos t)}{dt} \right\} \\ &= a(0 + \sin t) \\ &= a \sin t\end{aligned}$$

Now,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{a(1 + \cos t)}{a \sin t} \\ &= \frac{2 \cos^2 t/2}{2 \sin t/2 \cos t/2} \\ &= \cot t/2.\end{aligned}$$

Example 11. Find $\frac{dy}{dx}$ if $x^2 + y^2 = \sin xy$.

Solution

Given,

$$x^2 + y^2 = \sin xy$$

Differentiating both sides with respect to x, we get,

$$\frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} (\sin xy)$$

$$\text{or, } \frac{d}{dx} (x^2) + \frac{d}{dx} (y^2) = \frac{d(\sin xy)}{d(xy)} \cdot \frac{d(xy)}{dx}$$

$$\text{or, } 2x + \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = \cos xy \cdot \left\{ x \frac{d(dy)}{dx} + y \frac{d(dx)}{dx} \right\}$$

$$\text{or, } 2x + 2y \frac{dy}{dx} = \cos xy \left(x \frac{dy}{dx} + y \right)$$

$$\text{or, } 2x + 2y \frac{dy}{dx} = \cos xy \frac{dy}{dx} + y \cos xy$$

$$\text{or, } 2x - y \cos xy = x \cos xy \frac{dy}{dx} - 2y \frac{dy}{dx}$$

$$\text{or, } (x \cos xy - 2y) \frac{dy}{dx} = 2x - y \cos xy$$

$$\therefore \frac{dy}{dx} = \frac{2x - y \cos xy}{x \cos xy - 2y}.$$

Example 12. Find $\frac{dy}{dx}$ of $y = e^{\sin(\log x)}$.

Solution

Here, $y = e^{\sin(\log x)}$

Differentiating both sides with respect to x , we get,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [e^{\sin(\log x)}] \\ &= \frac{d[e^{\sin(\log x)}]}{d[\sin(\log x)]} \times \frac{d[\sin(\log x)]}{d(\log x)} \times \frac{d(\log x)}{dx} \\ &= e^{\sin(\log x)} \cdot \cos(\log x) \cdot \frac{1}{x}\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x} \cdot \cos(\log x) \cdot e^{\sin(\log x)}$$

Example 13. Find $\frac{dy}{dx}$ of $y = e^{5x} \cdot \sin(\log x)$.

Solution

We have,

$$y = e^{5x} \cdot \sin(\log x)$$

Differentiating both sides with respect to x , we get,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \{e^{5x} \cdot \sin(\log x)\} \\ &= e^{5x} \times \frac{d}{dx} \sin(\log x) + \sin(\log x) \cdot \frac{d}{dx} (e^{5x}) \\ &= e^{5x} \times \frac{d \sin(\log x)}{d(\log x)} \times \frac{d(\log x)}{dx} + \sin(\log x) \times \frac{d(e^{5x})}{d(5x)} \times \frac{d(5x)}{dx} \\ &= e^{5x} \times \cos(\log x) \times \frac{1}{x} + \sin(\log x) \times e^{5x} \times 5 \\ &= \frac{1}{x} \cdot e^{5x} \cos(\log x) + 5 e^{5x} \sin(\log x)\end{aligned}$$

$$\therefore \frac{dy}{dx} = e^{5x} \left[\frac{1}{x} \cos(\log x) + 5 \sin(\log x) \right]$$

Example 14. Find $\frac{dy}{dx}$ of $x^y = e^{x-y}$.

Solution

Here,

$$x^y = e^{x-y}$$

Taking logarithms on both sides, we get,

$$\ln x^y = \ln e^{x-y}$$

$$\text{or, } y \ln x = x - y$$

$$\text{as } \ln e = 1$$

$$\text{or, } y(1 + \ln x) = x$$

Differentiating both sides w.r.t. ' x ', we get,

$$\frac{d[y(1 + \ln x)]}{dx} = \frac{dx}{dx}$$

$$\text{or, } \frac{yd(1 + \ln x)}{dx} + (1 + \ln x) \frac{dy}{dx} = 1$$

$$\text{or, } y\left(0 + \frac{1}{x}\right) + (1 + \ln x) \frac{dy}{dx} = 1$$

$$\text{or, } (1 + \ln x) \frac{dy}{dx} = 1 - \frac{y}{x}$$

$$\text{or, } (1 + \ln x) \frac{dy}{dx} = \frac{x - y}{x} = \frac{y \ln x}{x} \quad [\because x - y = y \ln x]$$

$$\therefore \frac{dy}{dx} = \frac{y \ln x}{x(1 + \ln x)} = \frac{\ln x}{(1 + \ln x)^2}$$

$$= \frac{\ln x}{(\ln e + \ln x)^2}$$

$$= (\ln x)(\ln ex)^{-2} \left[\because \frac{y}{x} = \frac{1}{1 + \ln x} \right]$$

Example 15. If $y = x^4 - 5x^2 + 3x + 21$, then find $\frac{d^2y}{dx^2}$.

Solution

We have,

$$y = x^4 - 5x^2 + 3x + 21$$

Differentiating both sides w.r.t. ' x ', we have,

$$\frac{dy}{dx} = \frac{d}{dx} (x^4 - 5x^2 + 3x + 21)$$

$$\frac{dy}{dx} = 4x^3 - 10x + 3$$

Differentiating w.r. to x , we have,

$$\frac{d^2y}{dx^2} = \frac{d}{dx} (4x^3 - 10x + 3)$$

$$= 12x^2 - 10.$$

Exercise 2.2

Derivative of Algebraic Functions

1. Find the differential coefficient of

a. x^6

b. $2x^{1/2}$

c. $\frac{6}{\sqrt[3]{x^2}}$

d. $7x^{10} + 5x^3 - 6$

e. $ax^3 + bx^2 + cx + d$

f. $x^4 - 4 + x^{-2} - 5x^{-5}$

2. Find the differential coefficient of

a. $y = (x^2 + 5x)(3x^2 - x)$

c. $y = (x^2 + 7)(x^2 + 10)$

3. Find the derivative of

a. $y = \frac{x^2}{x-1}$

c. $y = \frac{3x^2 - 2}{x^2 + 7}$

e. $y = \frac{x^3 + 3x + 1}{x^2 + 1}$

4. Find the derivatives of

a. $y = (x+1)^6$

c. $y = \sqrt{ax^2 + bx + c}$

e. $y = \sqrt{\frac{1+x}{1-x}}$

5. Find derivatives of

a. $y = \frac{1}{\sqrt{x+b} - \sqrt{x}}$

c. $y = \frac{1}{\sqrt{x+a} - \sqrt{x+b}}$

6. Find $\frac{dy}{dx}$ if

a. $x = at^2, y = 2at$

b. $x = t^2 - 1, y = t^4 - 1$

c. $y = z^3 + 2z + 1, x = z^2 + 2$

d. $x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$

e. $x = \frac{3at}{1+t^2}, y = \frac{3at^2}{1+t^2}$

7. Find the derivative of y w.r.t. ' x ' in each of the following.

a. $x^2 + y^2 = 1$

b. $y^2 = 4ax$

c. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

d. $x^3 + y^3 = 3axy$

e. $(x^2 + y^2)^2 = xy$

b. $y = (x^2 + 1)(x^8 + 2)$

d. $y = (x^3 - 3x^2 + 4)(4x^3 + x^2 - 1)$

b. $y = \frac{4x}{x^2 + 1}$

d. $y = \frac{x^4 + 1}{x^2 + 1}$

b. $y = (4x^3 - 5x^2 + 1)^4$

d. $y = \frac{1}{\sqrt{a^2 - x^2}}$

f. $x\sqrt{1+x^2}$

b. $y = \frac{1}{\sqrt{3x-2} - \sqrt{3x-5}}$

Derivative of Trigonometric Functions

8. Find the derivatives of the following functions.

a. $\sin(px + q)$

b. $\cos^3 x$

c. $\tan^3(x^2)$

d. $\sec^3\left(\frac{3ax+b}{c}\right)$

e. $x^2 \tan x$

f. $(1+x^2) \sin x$

g. $\frac{x^2}{\sin 3x}$

h. $\frac{\sin \sqrt{x}}{\sqrt{x}}$

i. $\tan^2 \sin(\sqrt{x})$

j. $\sec^2(\tan \sqrt{x})$

k. $\sin 3x \cos x$

l. $\cos 6x \cos 2x$

m. $\sin \frac{1+x^2}{1-x^2}$

9. Find $\frac{dy}{dx}$ if

a. $x + y = \sin y$

b. $x - y = \sin xy$

c. $xy = \sin(x + y)$

d. $x^2 + y^2 = \sin xy$

e. $x^2 y^2 = \tan xy$

f. $x^2 y = \sec xy^2$

10. Find $\frac{dy}{dx}$, when

a. $x = a \cos^2 \theta, y = b \sin^2 \theta$

b. $x = 2a \tan \theta, y = a \sec^2 \theta$

c. $x = a(t + \sin t), y = a(1 - \cos t)$

d. $x = a(\cos t + t \sin t), y = a(\sin t - t \cos t)$

11. a. Find $\frac{dy}{dx}$ when $y = \sin \theta$ and $\theta = 5x^2 - 6x + 2$ b. Find the derivative of $\sin x$ w.r. to $\cos x$.c. Find the derivative of $\tan x$ w.r. to $\cot x$.

Derivative of Exponential and Logarithmic Functions

12. Find the derivatives of

a. $\ln(3x - 2)$

b. $\ln x^5$

c. $\ln(ax^2 + bx + c)$

d. $\ln(\sin x^2)$

c. $\ln(x + \tan x)$

g. $\ln(e^{\sin x} + e^{-\sin x})$

i. $\frac{\ln x}{\cos x}$

13. Find $\frac{dy}{dx}$, when

a. $xy = \ln(x+y)$

c. $x = e^{\cos t}, y = e^{\sin t}$

e. $x^y = y^x$

14. Find the derivatives of

a. $e^{\sin x}$

c. $(\sin x)^x$

Higher Order Derivatives

15. Find all successive derivatives of

a. $y = 2x^2 - x^3$

b. $y = x^4 + 3x^2 + 5$

Very Short Answer Questions

1. If $f(x) = 4 - x^2$ then find $f'(-3)$.2. If $y = t - \frac{1}{t}$ then find $\frac{dy}{dt}$.3. Find the derivative of $\sqrt{x}$.4. Find the derivative of $y = \frac{1}{x}$.5. If $y = t^3 + 2t + 1$ and $x = t^2 + 2$ then find $\frac{dy}{dx}$.6. If $x^2 + y^2 = 4$ then find $\frac{dy}{dx}$.7. Write the derivative of $\cos x$.8. Find the derivative of $x^2 - \sin x$.9. If $y = \sin x \cos x$ then find $\frac{dy}{dx}$.

f. $\ln(\ln x)$

h. $x \ln x$

b. $x^2 + y^2 = \ln(x^2 + y^2)$

d. $x = \ln t + \sin t, y = e^t + \cos t$

b. $e^{\sin(\ln x)}$

10. If $y = \sin^5 x$ then find $\frac{dy}{dx}$ at $x = \frac{\pi}{3}$.11. If $y = \cos t$ and $x = \sin t$ then find $\frac{dy}{dx}$.12. Find the derivative of $\sin x$ with respect to $\cos x$.13. Find the derivative of e^{2x} .14. What is the derivative of 5^x ?15. If $y = \ln t + e^t$ then find $\frac{dy}{dx}$.

Answers

1. a. $6x^5$ b. $\frac{1}{\sqrt{x}}$ c. $\frac{-4}{x^{5/2}}$ d. $70x^9 + 15x^2$

e. $3ax^2 + 2bx + c$ f. $4x^3 - 2x^{-3} + 25x^{-6}$

2. a. $12x^3 + 42x^2 - 10x$ b. $10x^9 + 8x^7 + 4x$ c. $4x^3 + 34x$

d. $32x^7 - 84x^6 + 85x^4 - 12x^3 - 3x^2 + 14x$

3. a. $\frac{x^2 - 2x}{(x-1)^2}$ b. $\frac{4(1-x^2)}{(x^2+1)^2}$ c. $\frac{46x}{(x^2+7)^2}$ d. $\frac{2x^5 + 4x^3 - 2x}{(x^2+1)^2}$

e. $\frac{x^4 - 2x + 3}{(x^2+1)^2}$

4. a. $6(x+1)^5$ b. $8x(4x^3 - 5x^2 + 1)^3(6x-5)$ c. $\frac{2ax+b}{2\sqrt{ax^2+bx+c}}$

d. $\frac{x}{(a^2-x^2)^{3/2}}$ e. $\frac{1}{\sqrt{1+x} \cdot (1-x)^{3/2}}$ f. $\frac{2x^2+1}{\sqrt{1+x^2}}$

5. a. $\frac{1}{2b} \left[\frac{1}{\sqrt{x+b}} + \frac{1}{\sqrt{x}} \right]$ b. $\frac{1}{2} \left[\frac{1}{\sqrt{3x-2}} + \frac{1}{\sqrt{3x-5}} \right]$

c. $\frac{1}{2(a-b)} \left[\frac{1}{\sqrt{x+a}} + \frac{1}{\sqrt{x+b}} \right]$

6. a. $\frac{1}{t}$ b. $2t^2$ c. $\frac{3z^2+2}{2z}$ d. $\frac{t^2+1}{t^2-1}$

e. $\frac{2t}{1-t^2}$

7. a. $-\frac{x}{y}$ b. $\frac{2a}{y}$ c. $\frac{-b^2x}{a^2y}$ d. $\frac{ay-x^2}{y^2-ax}$

e. $\frac{y-4x^3-4xy^2}{4x^2y+4y^3-x}$

8. a. $p \cos(px+q)$ b. $-3 \cos^2 x \sin x$ c. $6x \tan^2 x^2 \sec^2 x^2$

d. $\frac{9a}{c} \sec^3 \left(\frac{3ax+b}{c} \right) \tan \left(\frac{3ax+b}{c} \right)$ e. $x^2 \sec^2 x + 2x \tan x$

f. $(1+x^2) \cos x + 2x \sin x$

g. $\frac{2x \sin 3x - 3x^2 \cos 3x}{\sin^2 3x}$

h. $\frac{1}{2x^2} (\sqrt{x} \cos \sqrt{x} - \sin \sqrt{x})$

i. $\frac{1}{\sqrt{x}} \tan(\sin \sqrt{x}) \sec^2(\sin \sqrt{x}) \cos \sqrt{x}$

- j. $\frac{\sec^2(\tan \sqrt{x}) \tan(\tan \sqrt{x}) \sec^2 \sqrt{x}}{\sqrt{x}}$ k. $2 \cos 4x + \cos 2x$
- l. $-2 \sin 4x - 4 \sin 8x$ m. $\frac{4x}{(1-x^2)^2} \cos\left(\frac{1+x^2}{1-x^2}\right)$
9. a. $\frac{1}{\cos y - 1}$ b. $\frac{1-y \cos xy}{1+x \cos xy}$ c. $\frac{\cos(x+y)-y}{x-\cos(x+y)}$ d. $\frac{2x-y \cos xy}{x \cos xy - 2y}$
- e. $-\frac{1}{x}$ f. $\frac{2xy-y^2 \sec xy^2 \tan xy^2}{2xy \sec xy^2 \tan xy^2 - x^2}$ g. $\tan \frac{t}{2}$ h. $\tan t$
10. a. $-\frac{b}{a}$ b. $\tan \theta$ c. $-\cot x$ d. $-\tan^2 x$
11. a. $(10x-6) \cos(5x^2-6x+2)$ b. $\frac{5}{x}$ c. $\frac{2ax+b}{ax^2+bx+c}$ d. $2x \cot x^2$
12. a. $\frac{3}{3x-2}$ b. $\frac{5}{x}$ c. $\frac{a(e^{ax}-e^{-ax})}{e^{ax}+e^{-ax}}$ d. $1+\ln x$
- e. $\frac{1+\sec^2 x}{x+\tan x}$ f. $\frac{1}{x \ln x}$ g. $\frac{a(e^{ax}-e^{-ax})}{e^{ax}+e^{-ax}}$ h. $1+\ln x$
13. a. $\frac{\cos x + x \sin x \ln x}{x \cos^2 x}$ b. $\frac{x(1-x^2-y^2)}{y(x^2+y^2-1)}$ c. $-e^{\sin t - \cos t} \cdot \cot t$ d. $\frac{t(e^t - \sin t)}{1+t \cos t}$
- e. $\frac{1-xy-y^2}{x^2+xy-1}$ f. $\frac{x(1-x^2-y^2)}{y(x^2+y^2-1)}$ g. $\frac{t(e^t - \sin t)}{1+t \cos t}$ h. $1+\ln x$
14. a. $e^{\sin x} \cos x$ b. $\frac{e^{\sin(x)} \cos(\ln x)}{x}$ c. $(\sin x)^x (x \cot x + \ln \sin x)$
15. a. $y_1 = 4x - 3x^2$
 $y_2 = 4 - 6x$
 $y_3 = -6$
 $y_4 = y_5 = \dots = 0$

Very Short Answer Questions

1. 6 2. $1 + \frac{1}{x}$ 3. $\frac{1}{2\sqrt{x}}$ 4. $-\frac{1}{x^2}$
5. $\frac{3x^2+2}{2x}$ 6. $-\frac{x}{y}$ 7. $-\sin x$ 8. $2x - \cos x$
9. $\cos^2 x - \sin^2 x$ 10. $\frac{45}{32}$ 11. $-\tan t$ 12. $-\cot x$
13. $2e^{2x}$ 14. $5^x \cdot \log 5$ 15. $\frac{1}{t} + e^t$

2.11 Derivatives of Inverse Circular Functions

In this section, we recall the some important results of inverse circular functions.

1. Self adjust property

- a. $\sin \sin^{-1} x = x = \sin^{-1} \sin x$
 b. $\cos \cos^{-1} x = x = \cos^{-1} \cos x$

c. $\tan \tan^{-1} x = x = \tan^{-1} \tan x$

2. Reciprocal property

- a. $\sin^{-1} x = \operatorname{cosec}^{-1} \frac{1}{x}$
 b. $\cos^{-1} x = \sec^{-1} \frac{1}{x}$
 c. $\tan^{-1} x = \cot^{-1} \frac{1}{x}$

3. Conversion property

- a. $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2}$
 b. $\sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1-x^2}}$
 c. $\sin^{-1} x = \sec^{-1} \frac{1}{\sqrt{1-x^2}}$
 d. $\sin^{-1} x = \cot^{-1} \frac{\sqrt{1-x^2}}{x}$

4. For any numerical value of x,

- a. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$
 b. $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$
 c. $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

5. For given numerical value of x.

- a. $\sin^{-1}(-x) = -\sin^{-1} x$
 b. $\cos^{-1}(-x) = \pi - \cos^{-1} x$
 c. $\tan^{-1}(-x) = -\tan^{-1} x$
 d. $\cot^{-1}(-x) = \pi - \cot^{-1} x$

6. $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right), xy < 1$

7. $\sin^{-1} x + \sin^{-1} y = \sin^{-1} \{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

8. $\cos^{-1} x + \cos^{-1} y = \cos^{-1} (xy - \sqrt{1-x^2} \sqrt{1-y^2})$

Now, we study about the derivative of inverse circular functions.

1. Derivative of $\sin^{-1} x$ by first principles

$$\text{Let } f(x) = \sin^{-1} x$$

$$\therefore f(x+h) = \sin^{-1}(x+h)$$

We have,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore \frac{d}{dx}(\sin^{-1} x) = \lim_{h \rightarrow 0} \frac{\sin^{-1}(x+h) - \sin^{-1} x}{h} \quad \dots(i)$$

$$\text{Put } y = \sin^{-1} x \text{ and } y+k = \sin^{-1}(x+h)$$

$$\text{so that } x = \sin y \text{ and } x+h = \sin(y+k)$$

$$\text{Then, } h = \sin(y+k) - \sin y. \text{ As } h \rightarrow 0, k \rightarrow 0$$

Then from (i)

$$\frac{d}{dx}(\sin^{-1} x) = \lim_{h \rightarrow 0} \frac{y+k-y}{h}$$

$$= \lim_{k \rightarrow 0} \left\{ \frac{k}{\sin(y+k) - \sin y} \right\}$$

$$= \lim_{k \rightarrow 0} \left\{ \frac{k}{2 \cos\left(\frac{y+k+y}{2}\right) \sin\left(\frac{y+k-y}{2}\right)} \right\}$$

$$= \lim_{k \rightarrow 0} \left\{ \frac{k}{2 \cos\left(y + \frac{k}{2}\right) \cdot \sin \frac{k}{2}} \right\}$$

$$= \lim_{k \rightarrow 0} \left\{ \frac{1}{2 \cos\left(y + \frac{k}{2}\right) \cdot \frac{\sin \frac{k}{2}}{\frac{k}{2} \times 2}} \right\}$$

$$= \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

Next, we find the derivative of inverse circular functions using differentiation formulas.

1. Derivative of $\sin^{-1} x$

$$\text{Let } y = \sin^{-1} x$$

$$\text{Then, } \sin y = x$$

Differentiating both sides w.r.to x , we have,

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

$$\text{or, } \frac{d(\sin y)}{dy} \cdot \frac{dy}{dx} = 1$$

$$\text{or, } \cos y \cdot \frac{dy}{dx} = 1$$

$$\text{or, } \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\text{or, } \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

Similarly, we have

$$2. \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1.$$

$$3. \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

$$4. \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

$$5. \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.$$

$$6. \frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.$$

Illustrative Examples

Example 1. Find from definition, the derivative of $\ln \cos^{-1} x$.

Solution

$$\text{Let } f(x) = \ln \cos^{-1} x$$

$$\text{We have, } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore \frac{d}{dx}(\ln \cos^{-1} x) = \lim_{h \rightarrow 0} \frac{\ln \cos^{-1}(x+h) - \ln \cos^{-1} x}{h} \quad \dots(i)$$

Put $\cos^{-1} x = y$ and $\cos^{-1}(x+h) = y+k$
 then $x = \cos y$ and $x+h = \cos(y+k)$
 $\therefore h = \cos(y+k) - \cos y$
 Also, as $h \rightarrow 0, k \rightarrow 0$

From (i)

$$\frac{d}{dx}(\ln \cos^{-1} x) = \lim_{h \rightarrow 0} \frac{\ln(y+k) - \ln y}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\ln\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{k \rightarrow 0} \left\{ \frac{\ln\left(1 + \frac{k}{y}\right)}{\frac{k}{y}} \cdot \frac{\frac{k}{y}}{h} \right\}$$

$$= \frac{1}{y} \lim_{k \rightarrow 0} \left\{ \frac{k}{\cos(y+k) - \cos y} \right\}$$

$$= \frac{1}{y} \lim_{k \rightarrow 0} \left\{ \frac{k}{2 \sin\left(\frac{y+k+y}{2}\right) \sin\left(\frac{y-k}{2}\right)} \right\}$$

$$= \frac{1}{y} \lim_{k \rightarrow 0} \left\{ \frac{-1}{\sin\left(y + \frac{k}{2}\right) \cdot \sin\left(\frac{k}{2}\right)} \right\}$$

$$= \frac{1}{y} \cdot \frac{-1}{\sin y} = \frac{-1}{\cos^{-1} x \sqrt{1-x^2}}$$

$$= \frac{-1}{\cos^{-1} x \sqrt{1-x^2}}$$

Example 2. Find the derivative of $\sin^{-1} x^2$.

Solution

Let $y = \sin^{-1} x^2$.

Differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1} x^2)$$

$$= \frac{d(\sin^{-1} x^2)}{d(x^2)} \cdot \frac{d(x^2)}{dx}$$

$$= \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$$

$$= \frac{2x}{\sqrt{1-x^4}}$$

Example 3. Find the derivative of $\tan^{-1} x^3$.

Solution

Let $y = \tan^{-1} x^3$.

Differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} x^3)$$

$$= \frac{d(\tan^{-1} x^3)}{d(x^3)} \cdot \frac{d(x^3)}{dx}$$

$$= \frac{1}{\sqrt{1+(x^3)^2}} \cdot 3x^2$$

$$= \frac{3x^2}{1+x^6}$$

Example 4. Find $\frac{dy}{dx}$ when $y = \sin^{-1}(3x - 4x^3)$.

Solution

Given,

$$y = \sin^{-1}(3x - 4x^3)$$

Put,

$$x = \sin \theta \quad \dots (i)$$

Then,

$$y = \sin^{-1}(3 \sin \theta - 4 \sin^3 \theta)$$

$$\text{or, } y = \sin^{-1}(\sin 3\theta)$$

$$\therefore y = 3\theta \quad \dots (ii)$$

From (i),

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\sin \theta) = \cos \theta$$

From (ii),

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(3\theta) = 3$$

Now, by Chain Rule, we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{3}{\cos \theta}$$

$$= \frac{3}{\sqrt{1-\sin^2 \theta}}$$

$$= \frac{3}{\sqrt{1-x^2}}$$

[$\because$ Using (i)]

$$\therefore \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$$

Example 5. Find $\frac{dy}{dx}$ where $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

Solution

Given,

$$y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Put

$$x = \tan \theta$$

... (i)

Then,

$$y = \tan^{-1}\left(\frac{2 \tan \theta}{1 - \tan^2 \theta}\right)$$

$$\text{or, } y = \tan^{-1}(\tan 2\theta)$$

$$\therefore y = 2\theta \quad \dots (ii)$$

From (i)

$$x = \tan \theta$$

$$\therefore \frac{dx}{d\theta} = \frac{d}{d\theta}(\tan \theta) = \sec^2 \theta$$

and from (ii) $y = 2\theta$

$$\therefore \frac{dy}{d\theta} = \frac{d}{d\theta}(2\theta) = 2$$

Now,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{2}{\sec^2 \theta}$$

$$= \frac{2}{1 + \tan^2 \theta}$$

$$= \frac{2}{1 + x^2}$$

Alternatively,

We have,

$$y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \quad \dots (i)$$

Suppose, $x = \tan \theta$

... (ii)

Now, (i) becomes

$$y = \tan^{-1}\left(\frac{2 \tan \theta}{1 - \tan^2 \theta}\right) = \tan^{-1}(\tan 2\theta)$$

$$\text{or, } y = 2\theta$$

$$\text{or, } y = 2 \tan^{-1} x \quad [\because \text{from (ii)}]$$

Differentiating both sides with respect to x ,

$$\frac{dy}{dx} = \frac{d}{dx}(2 \tan^{-1} x)$$

$$= 2 \frac{d}{dx}(\tan^{-1} x) = 2 \left(\frac{1}{1+x^2}\right)$$

$$\text{Hence, } \frac{dy}{dx} = \frac{2}{1+x^2}$$

Example 6. Find the derivative of $\sin^{-1}(1-2x^2)$.

Solution

$$\text{Let } y = \sin^{-1}(1-2x^2)$$

Differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{d}{dx}\{\sin^{-1}(1-2x^2)\}$$

$$= \frac{d\{\sin^{-1}(1-2x^2)\}}{d(1-2x^2)} \cdot \frac{d(1-2x^2)}{dx}$$

$$= \frac{1}{\sqrt{1-(1-2x^2)^2}} \cdot \left\{ \frac{d}{dx}(1) - 2 \cdot \frac{d}{dx}(x^2) \right\}$$

$$= \frac{1}{\sqrt{1-(1-2x^2)^2}} (-4x)$$

$$= \frac{1}{\sqrt{1-1+4x^2-4x^4}} (-4x)$$

$$= \frac{1}{\sqrt{4x^2(1-x^2)}} (-4x)$$

$$= \frac{1}{2x\sqrt{1-x^2}} (-4x)$$

$$= -\frac{2}{\sqrt{1-x^2}}$$

Example 7. Find the derivative of $\tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)$

Solution

$$\text{Let } y = \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right) = \tan^{-1}\left(\frac{2 \sin x \cos x}{1+2 \cos^2 x - 1}\right)$$

$$= \tan^{-1}\left(\frac{2 \sin x \cos x}{2 \cos^2 x}\right)$$

$$= \tan^{-1}\left(\frac{\sin x}{\cos x}\right)$$

$$= \tan^{-1}(\tan x)$$

$$= x$$

Now, differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(x) = 1$$

Example 8. Find the derivative of $\tan^{-1}(\sec x + \tan x)$.

Solution

$$\begin{aligned}
 \text{Let } y &= \tan^{-1}(\sec x + \tan x) \\
 &= \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) \\
 &= \tan^{-1}\left(\frac{1 + \sin x}{\cos x}\right) \\
 &= \tan^{-1}\left(\frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}\right) \\
 &= \tan^{-1}\left[\frac{(\cos \frac{x}{2} + \sin \frac{x}{2})^2}{(\cos \frac{x}{2} - \sin \frac{x}{2})(\cos \frac{x}{2} + \sin \frac{x}{2})}\right] \\
 &= \tan^{-1}\left(\frac{\cos \frac{x}{2} + \sin \frac{x}{2}}{\cos \frac{x}{2} - \sin \frac{x}{2}}\right) \\
 &= \tan^{-1}\left(\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}}\right) \\
 &= \tan^{-1}\left(\frac{\tan \frac{\pi}{4} + \tan \frac{x}{2}}{1 - \tan \frac{\pi}{4} \cdot \tan \frac{x}{2}}\right) \\
 &= \tan^{-1}\left\{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\} \\
 &= \frac{\pi}{4} + \frac{x}{2}
 \end{aligned}$$

Thus, $y = \frac{\pi}{4} + \frac{x}{2}$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{\pi}{4} + \frac{x}{2}\right) \\
 &= \frac{1}{2}
 \end{aligned}$$

Exercise 2.3

1. Find the derivatives of the following functions:

- $\sin^{-1} x^3$
- $\cos^{-1} \sqrt{x}$
- $\sin^{-1} (3x - 4x^3)$
- $\cos^{-1} (4x^3 - 3x)$
- $\tan^{-1} \left(\frac{2x}{1-x^2}\right)$
- $\tan^{-1} \left(\frac{\sin x}{1 + \cos x}\right)$
- $\cos^{-1} \left(\frac{1-x^2}{1+x^2}\right)$
- $\tan^{-1} \left(\frac{\sqrt{1+x^2}-1}{x}\right)$
- $\tan^{-1} \left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

2. Find the derivative of

- $\ln(\sin^{-1} x)$
- $\sin^{-1}(\cos x)$
- $\tan^{-1} x + \sin^{-1} \left(\frac{2x}{1+x^2}\right)$

3. a. Find the derivative of $\tan^{-1} x$ with respect to $\cot^{-1} x$.

b. Find the derivative of $\cos^{-1} \left(\frac{1-x^2}{1+x^2}\right)$ with respect to $\tan^{-1} \left(\frac{3x-x^3}{1-3x^2}\right)$.

Very Short Answer Questions

- Write the derivative of $y = \sin^{-1} x$.
- Write the formula of derivative of $\cot^{-1} x$.
- Find the derivative of $\sin^{-1} x^2$.

Answers

- $\frac{3x^2}{\sqrt{1-x^2}}$
 - $-\frac{1}{2\sqrt{x}\sqrt{1-x}}$
 - $\frac{3}{\sqrt{1-x^2}}$
 - $-\frac{3}{\sqrt{1-x^2}}$
 - $\frac{2}{1+x^2}$
 - $\frac{1}{2}$
 - $\frac{2}{1+x^2}$
 - $\frac{1}{2(1+x^2)}$
- $\frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}}$
 - -1
- -1
 - $\frac{2}{3}$

Very Short Answer Questions

- $\frac{1}{\sqrt{1-x^2}}$
- $-\frac{1}{1+x^2}$
- $\frac{2x}{\sqrt{1-x^2}}$

2.12 Hyperbolic Functions

The hyperbolic functions are a set of functions which are closely related to the trigonometric functions. The hyperbolic functions are related to the hyperbola.

The hyperbolic sine is defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \dots (i)$$

The hyperbolic cosine is defined as

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \dots (ii)$$

The functions defined in (i) and (ii) have properties similar to the trigonometric functions $\sin x$ and $\cos x$.

Therefore

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad x \in \mathbb{R}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \quad x \in \mathbb{R}$$

$$\operatorname{coth} x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad x \in \mathbb{R} - \{0\}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \quad x \in \mathbb{R}$$

Also,

$$\cosh x + \sinh x = e^x$$

$$\cosh x - \sinh x = e^{-x}$$

Also, we have

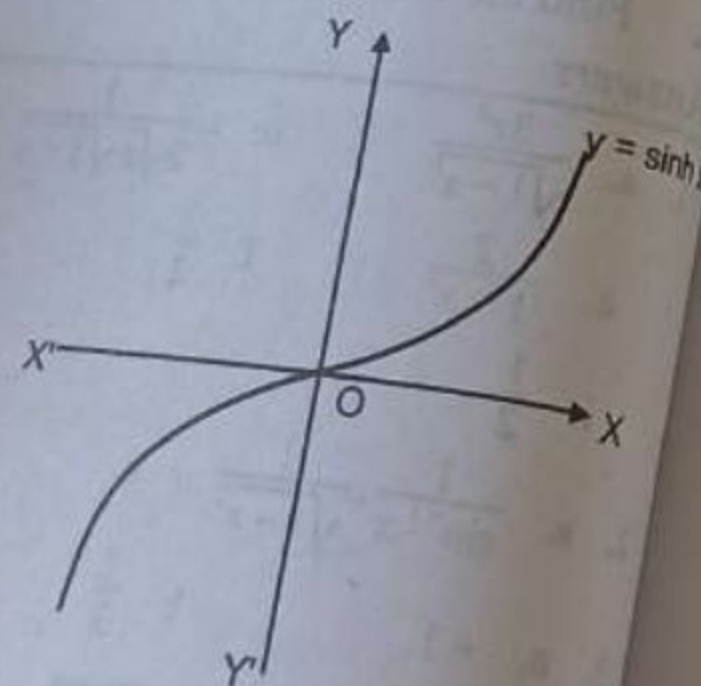
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

and

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\therefore \cosh x = \cos ix \quad (\because i^2 = -1)$$

$$\sinh x = -i \sin x$$



Moreover, $\sinh 0 = 0$ and $\cosh 0 = 1$

Also,

$$\sinh(-x) = -\sinh x,$$

$$\cosh(-x) = \cosh x$$

$$\text{and } \cosh^2 x - \sinh^2 x = 1$$

We also have

$$\lim_{x \rightarrow \infty} \sinh x = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{2} = \infty$$

$$\lim_{x \rightarrow -\infty} \sinh x = \lim_{x \rightarrow -\infty} \frac{e^x - e^{-x}}{2} = -\infty$$

Similarly, $\lim_{x \rightarrow \pm\infty} \cosh x = \infty$. The graph of $\sinh x$ and $\cosh x$ are shown in the above figures.

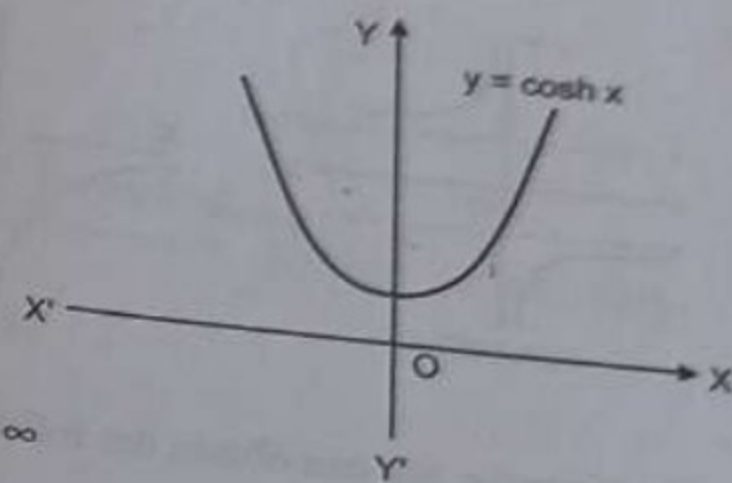
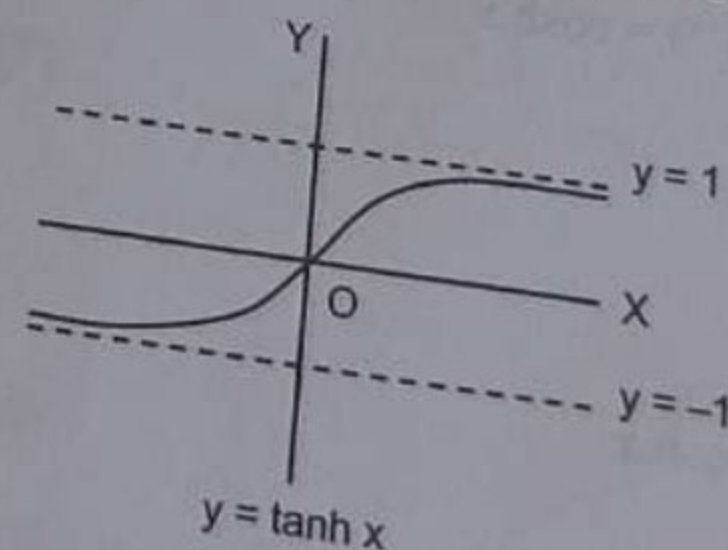
Also, $\tanh 0 = 0$ and $\tanh(-x) = -\tanh x$

Now

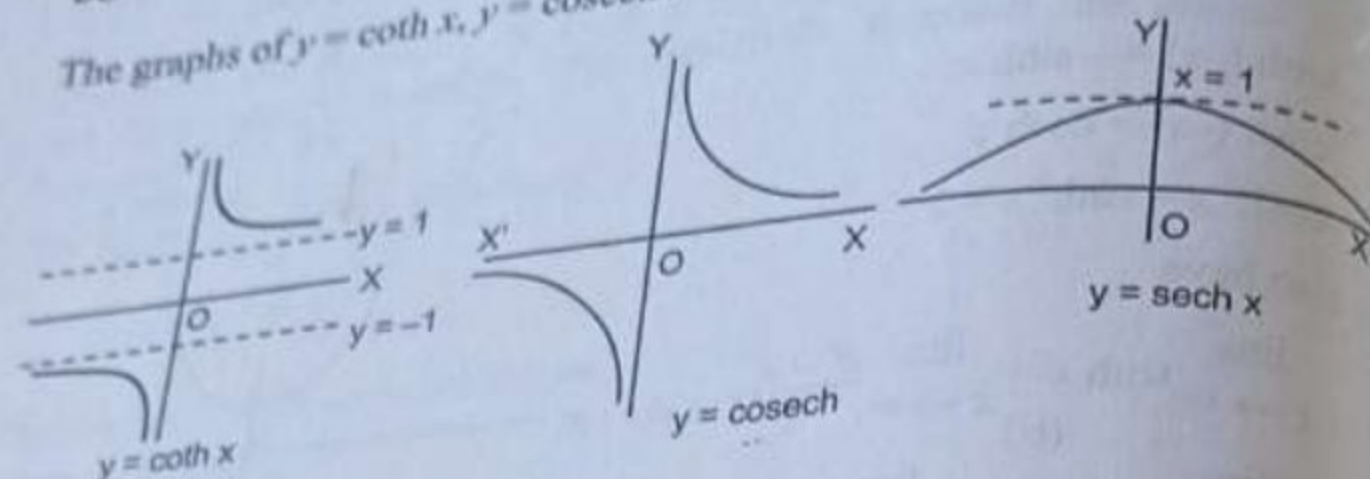
$$\begin{aligned} \lim_{x \rightarrow \infty} \tanh x &= \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ &= \lim_{x \rightarrow \infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} \\ &= 1. \end{aligned}$$

$$\begin{aligned} \text{and } \lim_{x \rightarrow -\infty} \tanh x &= \lim_{x \rightarrow -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ &= \lim_{x \rightarrow -\infty} \frac{e^{2x} - 1}{e^{2x} + 1} \\ &= -1. \end{aligned}$$

With these properties the graph of $y = \tanh x$ is as follows:



The graphs of $y = \coth x$, $y = \operatorname{cosech} x$ and $y = \operatorname{sech} x$ are shown below.



From above, we can obtain the following relations:

1. $\cosh^2 x + \sinh^2 x = \cosh 2x$
2. $\cosh^2 x - \sinh^2 x = 1$
3. $\sinh 2x = 2 \sinh x \cosh x$
4. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
5. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

2.13 Derivative of Hyperbolic Functions

Derivative of $\sinh x$

Let $y = \sinh x$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\sinh x) \\ &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) \\ &= \frac{1}{2} \frac{d}{dx}(e^x - e^{-x}) \\ &= \frac{1}{2}(e^x + e^{-x}) = \cosh x \\ &= \frac{e^x + e^{-x}}{2} \\ &= \cosh x\end{aligned}$$

Hence, $\frac{d}{dx}(\sinh x) = \cosh x$

Similarly, $\frac{d}{dx}(\cosh x) = \sinh x$.

Derivative of $\tanh x$

Let $y = \tanh x = \frac{\sinh x}{\cosh x}$. Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) \\ &= \frac{\cosh x \frac{d}{dx}(\sinh x) - \sinh x \frac{d}{dx}(\cosh x)}{(\cosh x)^2} \quad [\text{Using quotient rule}] \\ &= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} \\ &= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x\end{aligned}$$

Hence, $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

Similarly,

$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \cdot \tanh x$$

$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \cdot \coth x.$$

Derivative of Inverse Hyperbolic Functions

Derivative of $\sinh^{-1} x$

Let $y = \sinh^{-1} x$

Then $x = \sinh y$

Differentiating both sides w.r.t. y , we get

$$\begin{aligned}\frac{dx}{dy} &= \frac{d}{dy}(\sinh y) \\ &= \cosh y \\ &= \sqrt{1 + \sinh^2 y} \\ &= \sqrt{1 + x^2}\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}$$

Similarly, we can obtain

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}, (x > 1)$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, \quad (-1 < x < 1)$$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{1-x^2}, \quad (|x| > 1)$$

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}, \quad (0 < x < 1)$$

$$\text{and } \frac{d}{dx}(\operatorname{cosech}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}, \quad (x \in \mathbb{R} - \{0\}).$$

Illustrative Examples

Example 1. Find the derivative of $\ln \sinh x$.

Solution

$$\text{Let } y = \ln \sinh x$$

Differentiating both sides with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\ln \sinh x) \\ &= \frac{d(\ln \sinh x)}{d(\sinh x)} \cdot \frac{d(\sinh x)}{dx} \\ &= \frac{1}{\sinh x} \cdot \cosh x \\ &= \coth x \end{aligned}$$

Example 2. Find the derivative of $\operatorname{Arc} \tan \sinh x$.

Solution

$$\text{Let } y = \operatorname{Arc} \tan \sinh x$$

$$= \tan^{-1}(\sinh x)$$

Differentiating both sides with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\tan^{-1} \sinh x) \\ &= \frac{d(\tan^{-1} \sinh x)}{d(\sinh x)} \cdot \frac{d(\sinh x)}{dx} \\ &= \frac{1}{1 + \sinh^2 x} \cdot \cosh x \\ &= \frac{\cosh x}{\cosh^2 x} \\ &= \operatorname{sech} x \end{aligned}$$

Example 3. Find the derivative of $\tanh^{-1}(\sin x)$.

Solution

$$\text{Let, } y = \tanh^{-1}(\sin x)$$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \{ \tanh^{-1}(\sin x) \} \\ &= \frac{d \{ \tanh^{-1}(\sin x) \}}{d(\sin x)} \cdot \frac{d(\sin x)}{dx} \\ &= \left(\frac{1}{1 - \sin^2 x} \right) \cos x \\ &= \frac{\cos x}{\cos^2 x} \\ &= \frac{1}{\cos x} \\ &= \sec x \end{aligned}$$

Example 4. Find the derivative of $2 \tanh^{-1} \left(\tan \frac{x}{2} \right)$.

Solution

$$\text{Let } y = 2 \tanh^{-1} \left(\tan \frac{x}{2} \right)$$

Differentiating both sides with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left\{ 2 \tanh^{-1} \left(\tan \frac{x}{2} \right) \right\} \\ &= 2 \cdot \frac{d \left\{ \tanh^{-1} \left(\tan \frac{x}{2} \right) \right\}}{d \left(\tan \frac{x}{2} \right)} \cdot \frac{d \left(\tan \frac{x}{2} \right)}{d \left(\frac{x}{2} \right)} \cdot \frac{d \left(\frac{x}{2} \right)}{dx} \\ &= 2 \cdot \frac{1}{1 - \tan^2 \frac{x}{2}} \cdot \sec^2 \frac{x}{2} \cdot \frac{1}{2} \\ &= \frac{\sec^2 \frac{x}{2}}{\sin^2 \frac{x}{2}} \\ &= \frac{\sec^2 \frac{x}{2} \cdot \cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}} \\ &= \frac{1}{\cos x} \\ &= \sec x \end{aligned}$$

Example 5. Find the derivative of $e^{\sinh x}$.

Solution

Let $y = e^{\sinh x}$. Then

Differentiating both sides with respect to x , we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(e^{\sinh x}) \\ &= \frac{d(e^{\sinh x})}{d(\sinh x)} \cdot \frac{d(\sinh x)}{dx} \\ &= e^{\sinh x} \cdot \cosh x\end{aligned}$$

Example 6. Find the derivative of $x^{\cosh x}$.

Solution

Let $y = x^{\cosh x}$

Taking logarithm on both sides, we get

$$\ln y = \cosh x \ln x$$

Differentiating both sides with respect to x , we get

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\cosh x \ln x)$$

$$\text{or, } \frac{d}{dy}(\ln y) \cdot \frac{dy}{dx} = \cosh x \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(\cosh x)$$

$$\text{or, } \frac{1}{y} \cdot \frac{dy}{dx} = \cosh x \cdot \frac{1}{x} + \ln x \cdot \sinh x$$

$$\text{or, } \frac{dy}{dx} = y \left(\frac{1}{x} \cosh x + \ln x \sinh x \right)$$

$$\therefore \frac{dy}{dx} = x^{\cosh x} \left(\frac{1}{x} \cosh x + \ln x \sinh x \right)$$

Example 7. Find the derivative of $\left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right)^{nx}$.

Solution

$$\text{Let } y = \left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right)^{nx}$$

Taking logarithm on both sides, we get

$$\ln y = nx \ln \left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right)$$

Differentiating both sides with respect to x , we get

$$\frac{d}{dx}(\ln y) = \frac{d}{dx} \left\{ nx \ln \left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right) \right\}$$

$$\text{or, } \frac{d}{dy}(\ln y) \cdot \frac{dy}{dx} = nx \frac{d}{dx} \left\{ \ln \left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right) \right\} + \ln \left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right) \frac{d}{dx}(nx)$$

$$\text{or, } \frac{1}{y} \cdot \frac{dy}{dx} = nx \frac{d \left\{ \ln \left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right) \right\}}{d \left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right)} \cdot \frac{d \left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right)}{d \left(\frac{x}{a}\right)} \cdot \frac{d \left(\frac{x}{a}\right)}{dx} + \ln \left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right) \cdot n$$

$$\text{or, } \frac{1}{y} \cdot \frac{dy}{dx} = \left[nx \left\{ \frac{1}{\left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right)} \cdot \left(\cosh \frac{x}{a} + \sinh \frac{x}{a}\right) \cdot \frac{1}{a} \right\} + \ln \left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right) \cdot n \right]$$

$$\text{or, } \frac{dy}{dx} = y \left[\frac{nx}{a} + n \ln \left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right) \right]$$

$$\therefore \frac{dy}{dx} = n \left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right)^{nx} \left\{ \frac{x}{a} + \log \left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right) \right\}$$

Exercise 2.4

Find the derivatives of

1. $\ln(\sinh x)$
2. $\ln(\cosh 2x)$
3. $e^{\cosh x}$
4. $e^{\cosh^{-1} x}$
5. $\sinh^{-1} x - \operatorname{cosech}^{-1} x$
6. $\operatorname{Arc} \cosh(\sec x)$
7. $\tan^{-1}(\sinh x)$
8. $\cosh^{-1}(\sinh x)$
9. $x^{\cosh \frac{x}{a}}$
10. $\left(\cosh \frac{x}{a}\right)^{\ln x}$
11. $x^{\tanh \frac{x}{a}}$
12. $(\ln x)^{\sinh x}$

Very Short Answer Questions

1. Express $\sinh x$ in the terms of e^x and e^{-x} .
2. What is the derivative of $\cosh x$?
3. Write the derivative of $\operatorname{sech} x$.
4. What is the derivative of $\tanh^{-1} x$?

Answers

1. $\coth x$
2. $2 \tanh 2x$
3. $e^{\cosh x} \cdot \sinh x$
4. $e^{\cosh^{-1} x} \cdot \frac{1}{\sqrt{x^2 - 1}}$
5. $\frac{x+1}{x\sqrt{x^2+1}}$
6. $\sec x$
7. $\operatorname{sech} x$
8. $\frac{\cosh x}{\sqrt{\sinh^2 x - 1}}$
9. $x^{\cosh \frac{x}{a}} \left[\frac{\cosh \frac{x}{a}}{x} + \frac{\ln x \sinh \frac{x}{a}}{a} \right]$
10. $\left[\frac{1}{a} \ln x \tanh \frac{x}{a} + \frac{1}{x} \ln \cosh \frac{x}{a} \right] \left(\cosh \frac{x}{a} \right)^{\ln x}$
11. $x^{\tanh \frac{x}{a}} \left(\frac{1}{x} \tanh \frac{x}{a} + \frac{1}{a} \ln x \operatorname{sech}^2 \frac{x}{a} \right)$
12. $(\ln x)^{\sinh x} \left\{ \frac{\sinh x}{x \ln x} + \cosh x \cdot \ln(\ln x) \right\}$

Very Short Answer Questions

1. $\frac{e^x - e^{-x}}{2}$
2. $\sinh x$
3. $-\operatorname{sech} x \tanh x$
4. $\frac{1}{1-x^2}$

2.14 Partial Derivatives

In previous sections, we have studied the derivative of a function of a single independent variable. In this section, we deal on the derivatives of a function of several independent variables. While doing partial derivative, we differentiate a function with respect to one of the independent variable and we treat the remaining variables as constants and do the derivative as in the case of single variable.

Let $z = f(x, y)$. The partial derivative of z or f with respect to x is the rate at which z changes as x changes if y is held constant. In the same way, the partial derivative of z or f with respect to y is the rate at which z changes as y changes if x is held constant. Now we give the definition of partial derivative.

Let $z = f(x, y)$ be a function of two independent variables x and y . The partial derivative of z or f with respect to x , denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ of $f_x(x, y)$ or z_x or f_x , is defined as

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} \text{ provided that the limit exists.}$$

It is the ordinary derivative of f with respect to x keeping the other variable y as a constant.

In the same way, the partial derivative of $z = f(x, y)$ with respect to y is defined by

$$f_y(x, y) = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k} \text{ provided that the limit exists.}$$

Similarly, if $w = f(x, y, z)$ be a function of three independent variables x, y and z then the partial derivative of $w = f(x, y, z)$ with respect to x, y and z are respectively defined by

$$f_x(x, y, z) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$f_y(x, y, z) = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

$$f_z(x, y, z) = \lim_{\Delta z \rightarrow 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

These are called the first order partial derivatives. These methods of calculating partial derivatives are called **first principles** or **definition method**.

Note that the symbol ∂ (read as 'del') is used to denote partial derivative instead of d , the symbol used in ordinary derivative.

Example 1. Use definition of partial derivatives to find $f_x(x, y)$ and $f_y(x, y)$ for $f(x, y) = x^2y$

Solution

$$f(x, y) = x^2y$$

$$\text{Then, } f(x+h, y) = (x+h)^2y$$

$$\text{and } f(x, y+k) = x^2(y+k)$$

By definition, we have

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2y - x^2y}{h}$$

$$= \lim_{h \rightarrow 0} \frac{y\{(x+h)^2 - x^2\}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{y(x^2 + 2xh + h^2 - x^2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{y(2xh + h^2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{yh(2x + h)}{h}$$

$$= \lim_{h \rightarrow 0} y(2x + h)$$

$$= y(2x + 0)$$

$$= 2xy$$

Also, we have,

$$f_y(x, y) = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k}$$

$$= \lim_{k \rightarrow 0} \frac{x^2(y+k) - x^2y}{k}$$

$$= \lim_{k \rightarrow 0} \frac{x^2(y+k-y)}{k}$$

$$= \lim_{k \rightarrow 0} \frac{x^2k}{k}$$

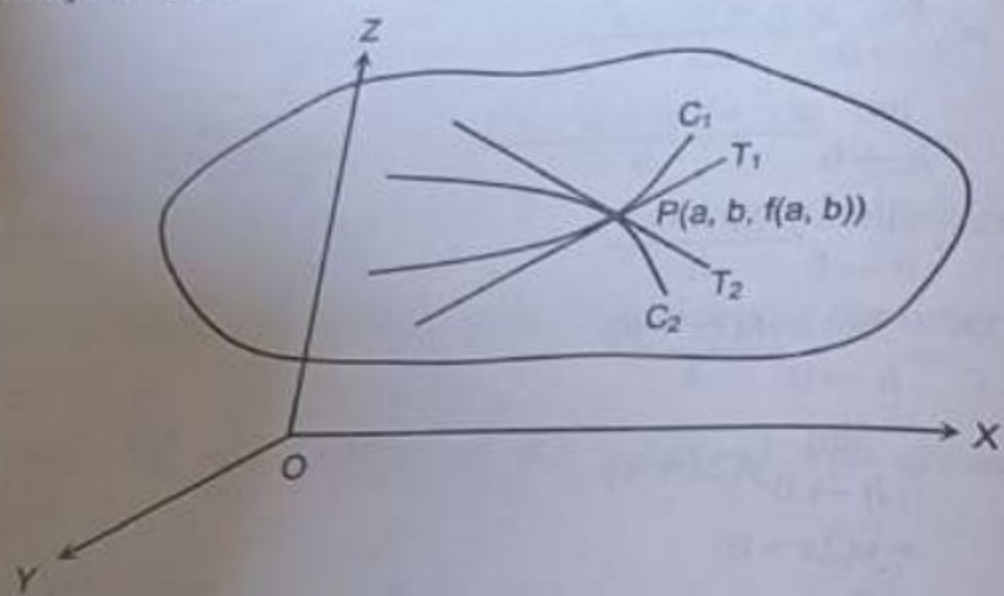
$$= \lim_{k \rightarrow 0} x^2$$

$$= x^2$$

2.14.1 Geometrical Interpretation of Partial Derivatives

Let $z = f(x, y)$ be a function of two independent variables x and y . The partial derivative $\frac{\partial z}{\partial x}$ at $(x, y) = (a, b)$ gives the slope of the tangent line drawn at (a, b) along the plane $y = b$ to the graph of $z = f(x, y)$. In the same way, the partial derivative $\frac{\partial z}{\partial y}$ at $(x, y) = (a, b)$ gives the slope of the tangent line drawn at (a, b) along the plane $x = a$ to the graph of $z = f(x, y)$.

Hence the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are interpreted geometrically as the slopes of curves of intersection of surface $z = f(x, y)$ by the planes $y = b$ and $x = a$ respectively.



2.14.2 Partial Derivatives at a Point

The partial derivative of $z = f(x, y)$ with respect to x at the point (a, b) is

$$\begin{aligned} \left. \frac{\partial f}{\partial x} \right|_{(a, b)} &= \left. \frac{d}{dx} f(x, b) \right|_{x=a} \\ &= \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h} \end{aligned}$$

provided that the limit exists.

It is denoted by

$$\left(\frac{\partial z}{\partial x} \right)_{(a, b)} \text{ or } f_x(a, b) \text{ or } z_x(a, b)$$

Similarly, the partial derivative of $z = f(x, y)$ with respect to y at the point (a, b) is

$$\begin{aligned} \left. \frac{\partial f}{\partial y} \right|_{(a, b)} &= \left. \frac{d}{dy} f(a, y) \right|_{y=b} \\ &= \lim_{k \rightarrow 0} \frac{f(a, b+k) - f(a, b)}{k} \end{aligned}$$

provided that the limit exists.
It is denoted by

$$\left(\frac{\partial z}{\partial y} \right)_{(a, b)} \text{ or } f_y(a, b) \text{ or } z_y(a, b)$$

Example 2. If $f(x, y) = x^2 + 3xy + 2y^2$ then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution

To find $\frac{\partial f}{\partial x}$, we treat y as a constant and differentiate f with respect to x .

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} (x^2 + 3xy + 2y^2) \\ &= 2x + 3y \cdot 1 + 0 \\ &= 2x + 3y \end{aligned}$$

[Treat y as a constant]

and

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} (x^2 + 3xy + 2y^2) \\ &= 0 + 3x \cdot 1 + 2 \cdot 2y \\ &= 3x + 4y \end{aligned}$$

Example 3. If $z = x^3 + 3xy^2 + y^3$ then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(2, 1)$.

Solution

Given,

$$z = x^3 + 3xy^2 + y^3$$

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial}{\partial x} (x^3 + 3xy^2 + y^3) \\ &= 3x^2 + 3y^2 \text{ (keeping } y \text{ fixed)} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} (x^3 + 3xy^2 + y^3) \\ &= 0 + 3x \cdot 2y + 3y^2 \text{ (keeping } x \text{ fixed)} \\ &= 6xy + 3y^2 \end{aligned}$$

Now,

$$\begin{aligned} \left(\frac{\partial z}{\partial x} \right)_{(2, 1)} &= \left(\frac{\partial z}{\partial x} \right)_{(2, 1)} \\ &= 3 \times 2^2 + 3 \times 1^2 \\ &= 15 \end{aligned}$$

$$\left(\frac{\partial z}{\partial x}\right)_{(2,1)} = \left(\frac{\partial z}{\partial x}\right)_{(2,1)} \\ = 6 + 2 \times 1 + 3 \times 1^2 \\ = 15$$

Second Order Partial Derivatives

If $z = f(x, y)$ is given function then the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also functions of x and y . Then their partial derivatives are called second order partial derivatives. They are

- i. $\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right)$, denoted by $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}
- ii. $\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$, denoted by $\frac{\partial^2 f}{\partial y \partial x}$ or f_{xy}
- iii. $\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$, denoted by $\frac{\partial^2 f}{\partial x \partial y}$ or f_{yx}
- iv. $\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right)$, denoted by $\frac{\partial^2 f}{\partial y^2}$ or f_{yy}

The partial derivative f_{xx} and f_{yy} are called pure second order partial derivatives and f_{xy} and f_{yx} are called mixed second order partial derivatives.

In general, $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$ i.e. $f_{yx} \neq f_{xy}$.

The following theorem gives the condition for equality of f_{yx} and f_{xy} .

The Mixed Derivative Theorem

If $z = f(x, y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b) then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

NOTE $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ but not f_{xy} .

For, $f_{yx} = (f_y)_x$

$$= \frac{\partial}{\partial x} (f_y) \quad \left[\because f_y = \frac{\partial f}{\partial y} \right]$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}$$

Similarly, $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$ but not f_{yx} .

Example 4. Find f_{xx}, f_{yy}, f_{xy} and f_{yx} for the function $f(x, y) = 3x^3 y^2$.

Solution

Given

$$f(x, y) = 3x^3 y^2$$

$$f_x = \frac{\partial}{\partial x} (3x^3 y^2) = 9x^2 y^2$$

$$f_y = \frac{\partial}{\partial y} (3x^3 y^2) = 6x^3 y$$

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x} (f_x) = \frac{\partial}{\partial x} (9x^2 y^2) = 18xy^2$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y} (f_y) = \frac{\partial}{\partial y} (6x^3 y) = 6x^3$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y} (f_x) = \frac{\partial}{\partial y} (9x^2 y^2) = 18x^2 y$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x} (f_y) = \frac{\partial}{\partial x} (6x^3 y) = 6x^3$$

Example 5. If $f(x, y) = x^3 e^{2y}$ then verify that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

Solution

Given,

$$f(x, y) = x^3 e^{2y}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^3 e^{2y}) = 3x^2 e^{2y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^3 e^{2y}) = x^3 2e^{2y} = 2x^3 e^{2y}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (3x^2 e^{2y}) = 3x^2 \cdot 2e^{2y} = 6x^2 e^{2y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (2x^3 e^{2y}) = 6x^2 e^{2y}$$

$$\therefore \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Example 6. If $z = \ln(x^2 + y^2)$ then show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

Solution

Given,

$$z = \ln(x^2 + y^2)$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \{ \ln(x^2 + y^2) \} = \frac{\partial \{ \ln(x^2 + y^2) \}}{\partial (x^2 + y^2)} \cdot \frac{\partial (x^2 + y^2)}{\partial x} = \frac{1}{x^2 + y^2} \cdot 2x$$

$$= \frac{2x}{x^2 + y^2}$$

$$\begin{aligned}\left(\frac{\partial z}{\partial y}\right) \text{ at } (2, 1) &= \left(\frac{\partial z}{\partial y}\right)_{(2, 1)} \\ &= 6 \times 2 \times 1 + 3 \times 1^2 \\ &= 15\end{aligned}$$

Second Order Partial Derivatives

If $z = f(x, y)$ is given function then the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also functions of x and y . Then their partial derivatives are called second order partial derivatives. They are

- $\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right)$, denoted by $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}
- $\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$, denoted by $\frac{\partial^2 f}{\partial y \partial x}$ or f_{xy}
- $\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$, denoted by $\frac{\partial^2 f}{\partial x \partial y}$ or f_{yx}
- $\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right)$, denoted by $\frac{\partial^2 f}{\partial y^2}$ or f_{yy}

The partial derivative f_{xx} and f_{yy} are called pure second order partial derivatives and f_{xy} and f_{yx} are called mixed second order partial derivatives.

In general, $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$ i.e. $f_{yx} \neq f_{xy}$.

The following theorem gives the condition for equality of f_{yx} and f_{xy} .

The Mixed Derivative Theorem

If $z = f(x, y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b) then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

NOTE $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ but not f_{xy} .

$$\text{For, } f_{yx} = (f_y)_x$$

$$= \frac{\partial}{\partial x} (f_y)$$

$$\left[\because f_x = \frac{\partial f}{\partial x} \right]$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}$$

Similarly, $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$ but not f_{yx} .

Example 4. Find f_{xx}, f_{yy}, f_{xy} and f_{yx} for the function $f(x, y) = 3x^3 y^2$. Derivatives UNIT 2 109

Solution

Given

$$f(x, y) = 3x^3 y^2$$

$$f_x = \frac{\partial}{\partial x} (3x^3 y^2) = 9x^2 y^2$$

$$f_y = \frac{\partial}{\partial y} (3x^3 y^2) = 6x^3 y$$

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x} (f_x) = \frac{\partial}{\partial x} (9x^2 y^2) = 18xy^2$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y} (f_y) = \frac{\partial}{\partial y} (6x^3 y) = 6x^3$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y} (f_x) = \frac{\partial}{\partial y} (9x^2 y^2) = 18x^2 y$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x} (f_y) = \frac{\partial}{\partial x} (6x^3 y) = 6x^3$$

Example 5. If $f(x, y) = x^3 e^{2y}$ then verify that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

Solution

Given,

$$f(x, y) = x^3 e^{2y}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^3 e^{2y}) = 3x^2 e^{2y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^3 e^{2y}) = x^3 2e^{2y} = 2x^3 e^{2y}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (3x^2 e^{2y}) = 3x^2 \cdot 2e^{2y} = 6x^2 e^{2y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (2x^3 e^{2y}) = 6x^2 e^{2y}$$

$$\therefore \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Example 6. If $z = \ln(x^2 + y^2)$ then show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

Solution

Given,

$$z = \ln(x^2 + y^2)$$

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x} \{ \ln(x^2 + y^2) \} = \frac{\partial \{ \ln(x^2 + y^2) \}}{\partial (x^2 + y^2)} \cdot \frac{\partial (x^2 + y^2)}{\partial x} \\ &= \frac{2x}{x^2 + y^2}\end{aligned}$$

$$\begin{aligned} \text{and} \\ \frac{\partial z}{\partial y} &= \frac{\partial}{\partial y} \{\ln(x^2 + y^2)\} \\ &= \frac{\partial \{\ln(x^2 + y^2)\}}{\partial(x^2 + y^2)} \cdot \frac{\partial(x^2 + y^2)}{\partial y} \\ &= \frac{1}{x^2 + y^2} \cdot 2y \\ &= \frac{2y}{x^2 + y^2} \end{aligned}$$

$$\begin{aligned} \text{Now,} \\ \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{2x}{x^2 + y^2} \right) \\ &= \frac{(x^2 + y^2) \frac{\partial}{\partial x} (2x) - 2x \frac{\partial}{\partial x} (x^2 + y^2)}{(x^2 + y^2)^2} \\ &= \frac{(x^2 + y^2) \cdot 2 - 2x \cdot 2x}{(x^2 + y^2)^2} \\ &= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2} \end{aligned}$$

$$\begin{aligned} \text{Again,} \\ \frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) \\ &= \frac{\partial}{\partial y} \left(\frac{2y}{x^2 + y^2} \right) \\ &= \frac{(x^2 + y^2) \frac{\partial}{\partial y} (2y) - 2y \frac{\partial}{\partial y} (x^2 + y^2)}{(x^2 + y^2)^2} \\ &= \frac{(x^2 + y^2) \cdot 2 - 2y \cdot 2y}{(x^2 + y^2)^2} \\ &= \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} \end{aligned}$$

$$\begin{aligned} \text{Now,} \\ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} &= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} \\ &= \frac{2y^2 - 2x^2 + 2x^2 - 2y^2}{(x^2 + y^2)^2} \\ &= \frac{0}{(x^2 + y^2)^2} \\ &= 0. \end{aligned}$$

Exercise 2.5

Derivatives UNIT 2 111

- Find the first order partial derivatives of the following functions.
 - $f(x, y) = 5x^4y^5$
 - $f(x, y) = x^3 + x^2y^2 - 2y^3$
 - $f(x, y) = x \sin xy$
 - $f(x, y) = \sin^{-1}(3x + 4y)$
 - $f(x, y, z) = x^3y^5z^2$
 - $f(x, y) = ax^2 + 2hxy + by^2$
 - $f(x, y) = \sqrt{x^2 + y^2}$
 - $f(x, y) = \frac{3xy}{x + \cos y}$
 - $f(x, y) = \ln(2x + 5y)$
 - $f(x, y, z) = \ln(3x + y + 2z)$
- Find the second order partial derivatives of the following functions.
 - $f(x, y) = x^3y^4 + 3x^2y^2 - 1$
 - $f(x, y) = \sin(2x + 3y)$
 - $f(x, y) = x^5e^{3y}$
- Calculate all the first order partial derivatives of $f(x, y) = e^{-x} \sin(x + y)$.
- If $u = \ln(x^3 + y^3 + z^3 - 3xyz)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x + y + z}$.
 - If $z = \frac{x^2 + y^2}{x + y}$, then show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.
- If $z = \ln(x^2 + y^2)$, then show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.
 - If $z = \cos(3x + 4y)$, then show that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.
- If $f(x, y) = x^3y^2 + x^5 + y^3$, then find f_x and f_y at the point $(1, 2)$.
 - If $f(x, y) = x^4y^2 - x^3y^3$, then find f_{xx} and f_{xxx} at the point $(2, 1)$.
 - If $f(x, y, z) = \frac{x}{x + y + z}$, then find $f_x(-1, 1, 2)$ and $f_y(-1, 1, 2)$.

Very Short Answer Questions

- If $f(x, y) = x^2 + y^2$ then find f_x at $(1, 2)$.
- If $f(x, y) = xy + \sin x$ then find $\frac{\partial f}{\partial y}$.
- If $f(x, y) = x^3 + y^2$ then find f_{xx} .
- If $f(x, y) = x^3 + \sin y$ then find f_{yy} .

Answers

1. a. $f_x = 20x^3y^2, f_y = 25x^2y^3$
 b. $f_x = 3x^2 + 2xy^2, f_y = 2x^2y - 6y^3$
 c. $f_x = xy \cos xy + \sin xy, f_y = x^2 \cos xy$
 d. $f_x = \frac{3y \cos y}{(x + \cos y)^2}, f_y = \frac{3x^2 + 3x \cos y + 3y \sin y}{(x + \cos y)^2}$
 e. $f_x = \frac{2}{\sqrt{1 - (3x + 4y)^2}}, f_y = \frac{4}{\sqrt{1 - (3x + 4y)^2}}$
 f. $f_x = \frac{2}{2x + 5y}, f_y = \frac{5}{2x + 5y}$
 g. $f_x = 3x^2y^2z^2, f_y = 5x^3y^4z^2, f_z = 2x^3y^2z$
 h. $f_x = \frac{3}{3x + y + 2z}, f_y = \frac{1}{3x + y + 2z}, f_z = \frac{2}{2x + y + 2z}$
 2. a. $f_x = 6xy^4 + 6y^2, f_y = 12x^2y^3 + 12xy, f_{xy} = 12x^2y^2 + 12xy, f_{yx} = 12x^3y^2 + 6x^2$
 b. $f_x = -4 \sin(2x + 3y), f_y = -2 \sin(2x + 3y), f_{xy} = -6 \sin(2x + 3y), f_{yx} = -9 \sin(2x + 3y)$
 c. $f_x = 20x^2e^{2y}, f_y = 15x^4e^{2y}, f_{xy} = 15x^4e^{2y}, f_{yx} = 9x^5e^{3y}$
 3. a. $f_x = e^{-x} \{\cos(x + y) - \sin(x + y)\}, f_y = e^{-x} \cos(x + y)$
 $f_{xy} = -e^{-x} \sin(x + y), f_{yx} = -e^{-x} \sin(x + y)$
 $f_{xx} = -e^{-x} \{\sin(x + y) + \cos(x + y)\}, f_{xy} = -e^{-x} \{\sin(x + y) + \cos(x + y)\}$
 4. a. 8, 5 b. 36, 48 c. $\frac{3}{4}, \frac{1}{4}$

Very Short Answer Questions

1. 2 2. x 3. 6x 4. $-\sin y$

Applications of Derivatives

UNIT

3

3.1 Introduction

Derivatives have wide range of applications in a large number of disciplines such as science, engineering, economics, computer, commerce and so on. The derivative as the rate of change concept has many applications in science and technology as well as in commerce. We can use the derivative to obtain information about the shape of the graph of a function.

In this chapter, we will focus our study to find extreme values of functions, to determine and analyze the shapes of graphs, to understand marginal concept and competitive equilibrium of firm.

3.2 The Derivatives and Slope of the Curve

Let $y = f(x)$ be a given function. Let Δx be the small increment in x and Δy be the corresponding small increment in y . Then,

$$y + \Delta y = f(x + \Delta x)$$

$$\text{or, } \Delta y = f(x + \Delta x) - f(x)$$

Then, $\frac{\Delta y}{\Delta x}$ is called the **average rate of change** of y with respect to x . The **instantaneous rate of change** of a function $y = f(x)$ with respect to x is defined to be

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

It is the definition of derivative of $y = f(x)$ with respect to x . Thus,

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

On the other hand, the derivative of $y = f(x)$ with respect to x denoted by $f'(x)$ or $f'(x)$ is the slope of curve or slope of tangent at any point of the curve $y = f(x)$. Thus, the slope of the curve at $x = a$ is the derivative of the curve (curve) at $x = a$. This concept has been already studied in unit 2.

3.3 Increasing and Decreasing Function

The derivative can be used in describing the behaviour of function. Now, we discuss about increasing and decreasing functions. These concepts are useful for determining where a function achieves maximum or minimum values.

Increasing Function

A function $f(x)$ is said to be increasing function on (a, b) if for all $x_1, x_2 \in (a, b)$,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

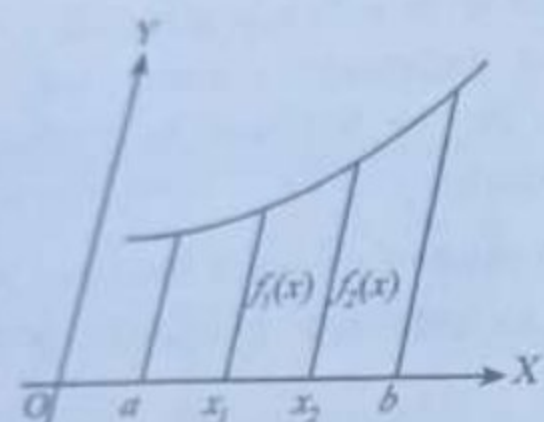


Fig. f is increasing on (a, b)

Decreasing Function

A function $f(x)$ is said to be decreasing function on (a, b) if for all $x_1, x_2 \in (a, b)$,

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

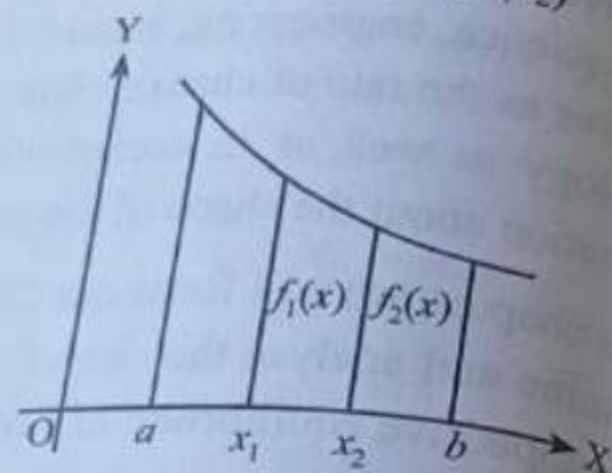


Fig. f is decreasing on (a, b)

The First Derivative Test for Increasing and Decreasing

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) .

- If $f' > 0$ at each point of (a, b) then f increases on $[a, b]$.
- If $f' < 0$ at each point of (a, b) then f decreases on $[a, b]$.

Thus functions with positive derivatives are increasing functions and the functions with negative derivatives are decreasing functions.

Example: Let us consider the function $f(x) = x^2$.

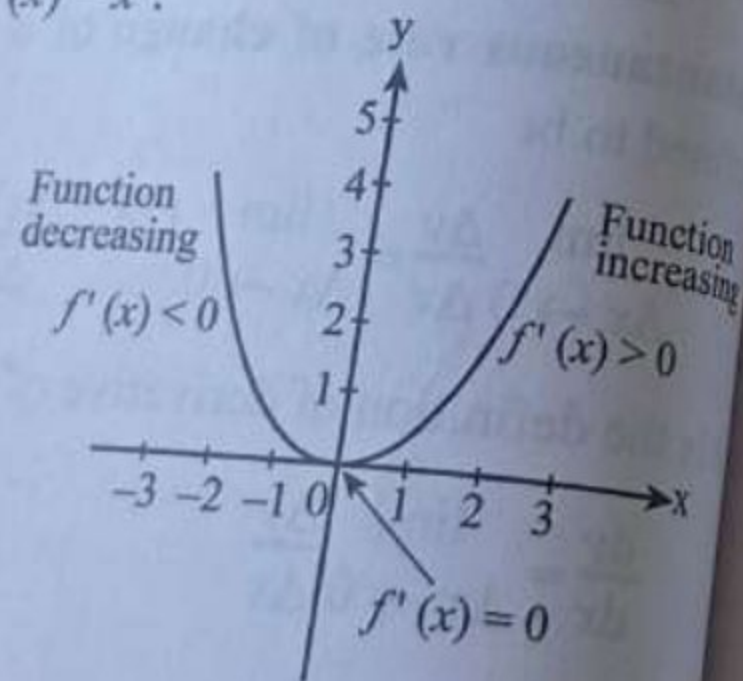
Then, $f'(x) = 2x$

Clearly $f'(x) = 2x > 0$ for all $x \in (0, \infty)$

$\therefore f$ is increasing on $(0, \infty)$.

Again, $f'(x) = 2x < 0$ for all $x \in (-\infty, 0)$

$\therefore f$ is decreasing on $(-\infty, 0)$.



NOTE

- A function that is increasing or decreasing on an interval I is called monotonic on I .
- A function is increasing (decreasing) on an interval then the function is increasing (decreasing) at every point within the interval.

3.4 Convexity of Curves

Geometrically, a function is said to be **concave up** (or **convex down**) on an interval if its curve bends upward and **concave down** (or **convex up**) on an interval if its curve bends downward.

Definition

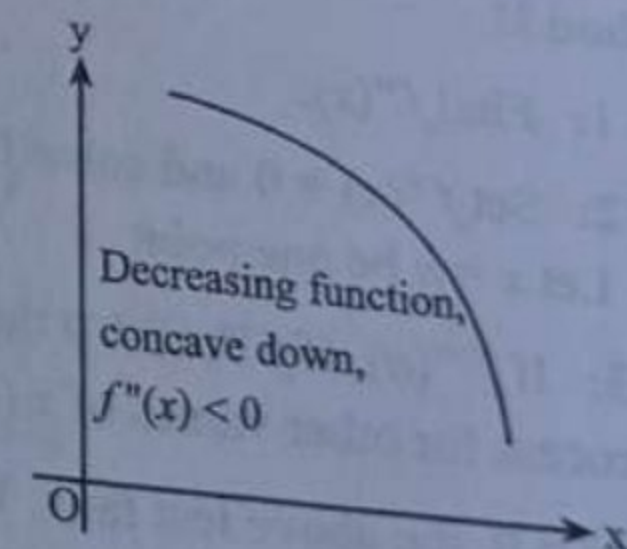
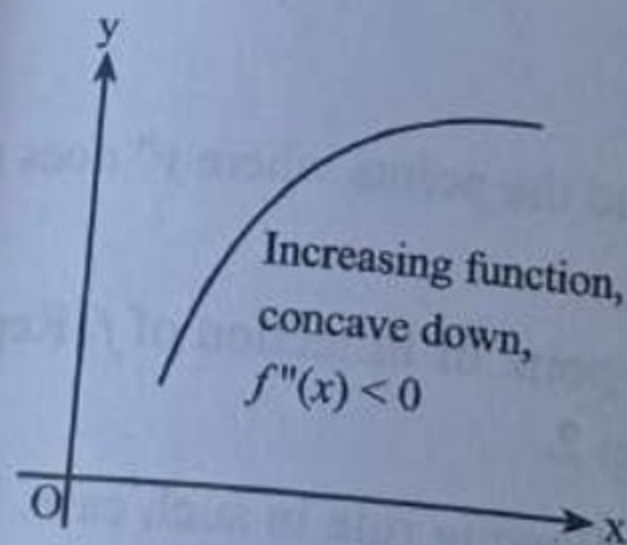
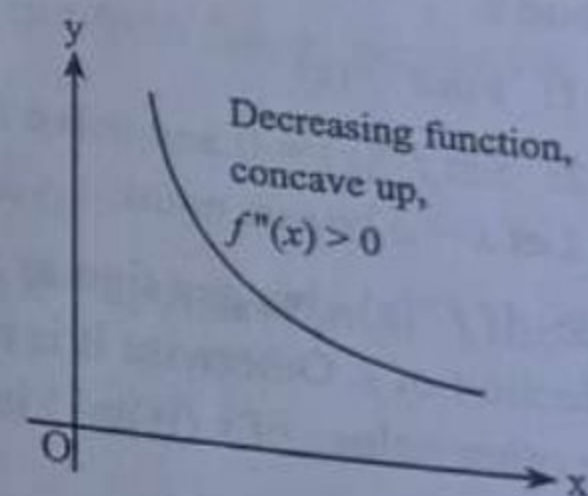
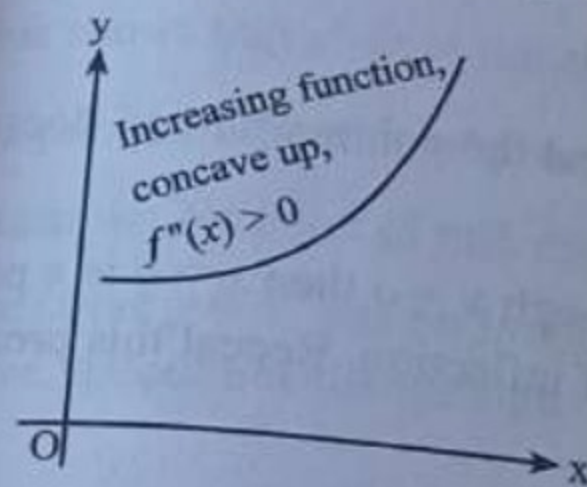
The graph of a differentiable function $y = f(x)$ is **concave up** (or **convex down**) on an interval where $f'(x)$ is increasing and **concave down** (or **convex up**) on an interval where $f'(x)$ is decreasing.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice differentiable on an interval I .

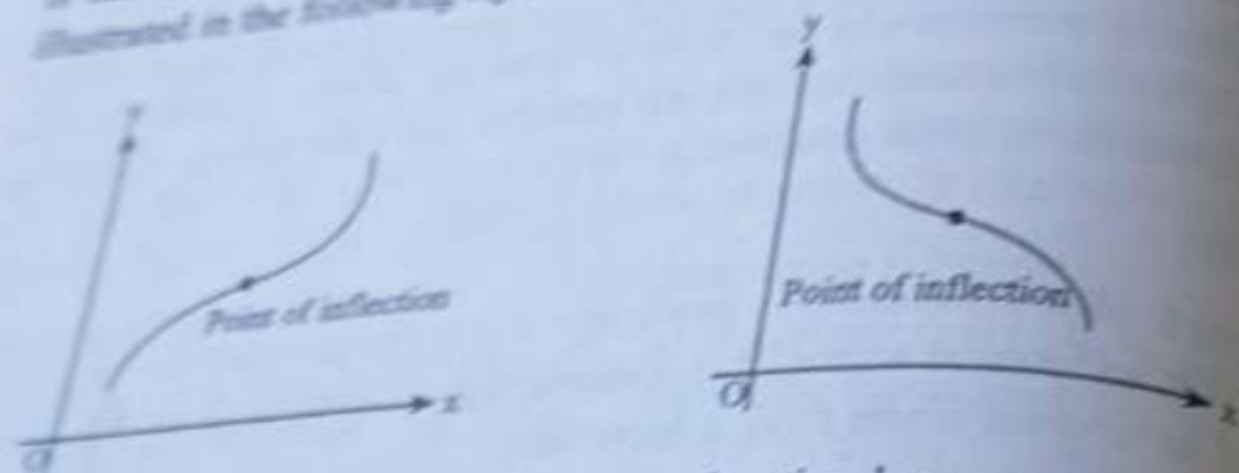
- If $f''(x) > 0$ on I , the graph of f over I is concave up.
- If $f''(x) < 0$ on I , the graph of f over I is concave down.

Note that the curve lies above the tangent lines are concave up whereas the curve that lies below the tangent lines are concave down.



Point of Inflection

A point on the graph of $y = f(x)$ at which the curve changes from concave downward to concave upward or from concave upward to concave downward is called point of inflection or inflection point. The inflection points are illustrated in the following figures.



To be precise, a point where the graph of a function has a tangent line and where the concavity changes is called a point of inflection.

Thus, a point of inflection on a curve $y = f(x)$ is a point where y'' is positive on one side and negative on the other. At such point y'' is either zero or undefined.

Now, we discuss the method for determining points of inflection.

Method I

Step 1: Find $f''(x)$.

Step 2: Set $f''(x) = 0$ and solve for x . Also find the points where y'' does not exist. Let $x = a$ be one point.

Step 3: If $f''(x)$ changes sign as x passes through $x = a$ then $x = a$ is a point of inflection of f . Otherwise it is not a point of inflection. Repeat this process for the other values of x (if any) in step 2.

Method II

Step 1: Find $f''(x)$.

Step 2: Set $f''(x) = 0$ and solve for x . Also find the points where y'' does not exist. Let $x = a$ be one point.

Step 3: If $f'''(a) \neq 0$ then $x = a$ is a point of inflection of f . Repeat this process for other values of x (if any) in step 2.

If $f'''(a) = 0$, the above test fails. We use the following rule in such case. Let $f''(a) = f'''(a) = \dots = f^{(n-1)}(a) = 0$ but $f^{(n)}(a) \neq 0$. If n is odd, f has a point of inflection at $x = a$. If n is even, f is concave up or down according as $f^{(n)}(a) > 0$ or < 0 .

Example: An inflection point where y'' does not exist.

The curve $y = x^{\frac{1}{3}}$ has a point of inflection at $x = 0$; but y'' does not exist there.

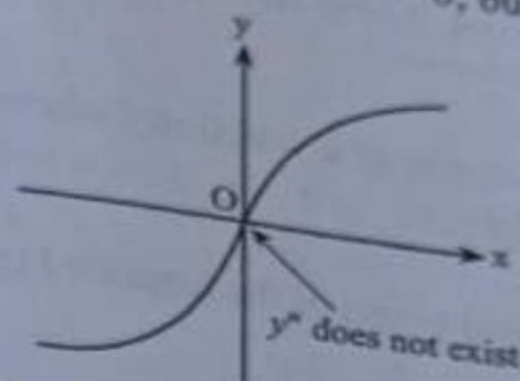


Fig. A point where y'' fails to exist can be a point of inflection

$$y' = \frac{d}{dx} (x^{\frac{1}{3}})$$

$$= \frac{1}{3} x^{\frac{1}{3}-1}$$

$$= \frac{1}{3} x^{-\frac{2}{3}}$$

$$y'' = \frac{d}{dx} \left(\frac{1}{3} x^{-\frac{2}{3}} \right)$$

$$= \frac{1}{3} \left(-\frac{2}{3} \right) x^{-\frac{2}{3}-1}$$

$$= -\frac{2}{9} x^{-\frac{5}{3}}$$

$$= -\frac{2}{9x^{\frac{5}{3}}}$$

This shows that y'' does not exist at $x = 0$. But from figure, we see that $x = 0$ is a point of inflection of $y = x^{\frac{1}{3}}$.

Example: No point of inflection where $y'' = 0$.

The curve $y = x^4$ has no inflection point at $x = 0$. Although $y'' = 12x^2$ is zero there, it does not change sign.

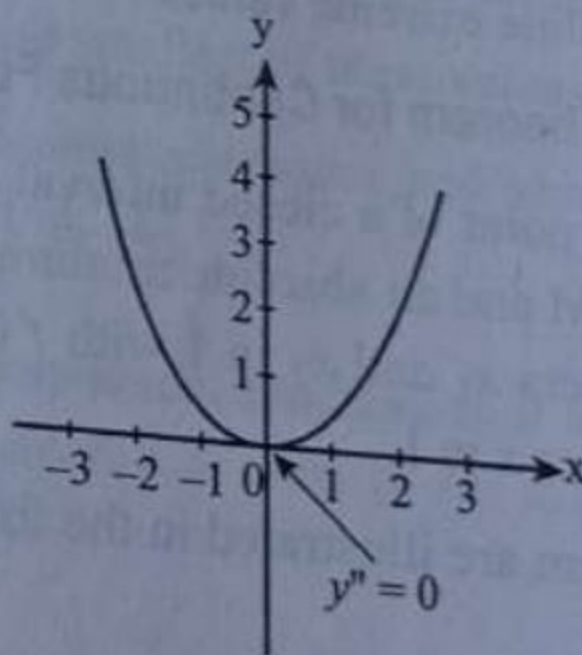


Fig. The graph of $y = x^4$ has no inflection point at $x = 0$ even though $y'' = 0$ there

Stationary Point

A point on the curve $y = f(x)$ where the tangent is parallel to the x -axis, known as the stationary point. At the stationary point, $y' = 0$.

Critical Point

An interior point of the domain of a function f where f' is zero or undefined is called a critical point of f .

Example 1. Find the critical points of the function $f(x) = x^{\frac{1}{3}}(x-4)$.

Solution

Here,

$$\begin{aligned} f(x) &= x^{\frac{1}{3}}(x-4) \\ &= x^{\frac{4}{3}} - 4x^{\frac{1}{3}} \\ f'(x) &= \frac{d}{dx} \left(x^{\frac{4}{3}} - 4x^{\frac{1}{3}} \right) \\ &= \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{-\frac{2}{3}} \\ &= \frac{4}{3}x^{-\frac{2}{3}}(x-1) \\ &= \frac{4(x-1)}{3x^{\frac{2}{3}}} \end{aligned}$$

Clearly, $f'(x)$ is zero at $x = 1$ and is undefined at $x = 0$.

$\therefore x = 0$ and $x = 1$ are the critical points of $f(x)$.

3.5 Maximization and Minimization of a Function

A function that is continuous at every point of a closed interval has an absolute maximum and an absolute minimum value on that interval. Now, we give a theorem about absolute extreme values.

Theorem (The Max. Min. Theorem for Continuous Functions)

If f is continuous at every point of a closed interval I , then f assumes both an absolute maximum value M and an absolute minimum value m somewhere in I . That is, there are numbers x_1 and x_2 in I with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in I .

The concepts of this theorem are illustrated in the following figures.

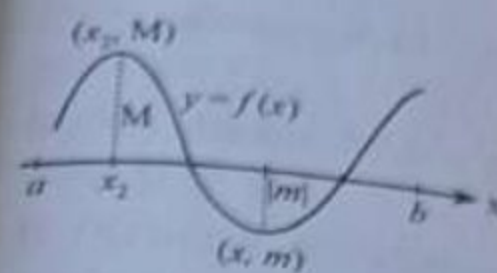


Fig.: Maximum and minimum at interior points

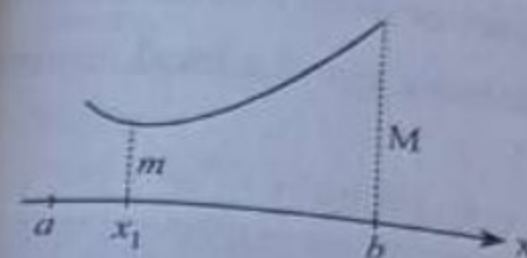


Fig.: Maximum at end point and minimum at interior point

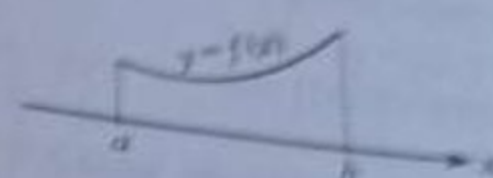


Fig.: Maximum and minimum at end points

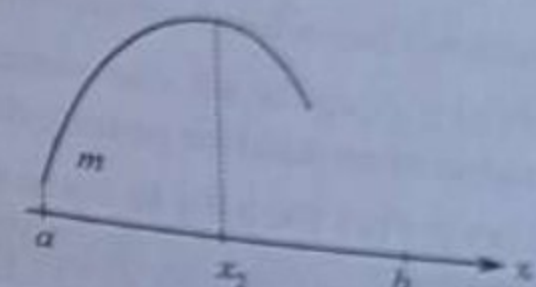


Fig.: Maximum at interior point and minimum at end point

Definition**Absolute Extreme Values**

Let f be a function defined on the domain D . Then the function f has an **absolute maximum value** on D at a point c if $f(x) \leq f(c)$ for all x in D ; and an **absolute minimum value** on D at c if $f(x) \geq f(c)$ for all x in D . Absolute extreme values are also called **global extreme values**.

Steps for finding absolute extreme values:

Step 1: Find all critical points.

Step 2: Evaluate f at all critical points and end points.

Step 3: Take the largest and smallest of these values.

Example 2. Find the absolute maximum and absolute minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution

Since $f(x) = x^2$ is a polynomial function, so it is differentiable everywhere in $\mathbb{R}$. Thus, f is differentiable at every points in $[-2, 1]$. So, the only critical point is where $f'(x) = 0$.

$$\text{or, } 2x = 0$$

$$\therefore x = 0.$$

Now, we have to evaluate f at critical point $x = 0$ and end points $x = -2$ and $x = 1$.

$$f(0) = 0^2 = 0$$

$$f(-2) = (-2)^2 = 4$$

$$f(1) = 1^2 = 1$$

Absolute maximum value = 4 at $x = 2$.

Absolute minimum value = 0 at $x = 0$.

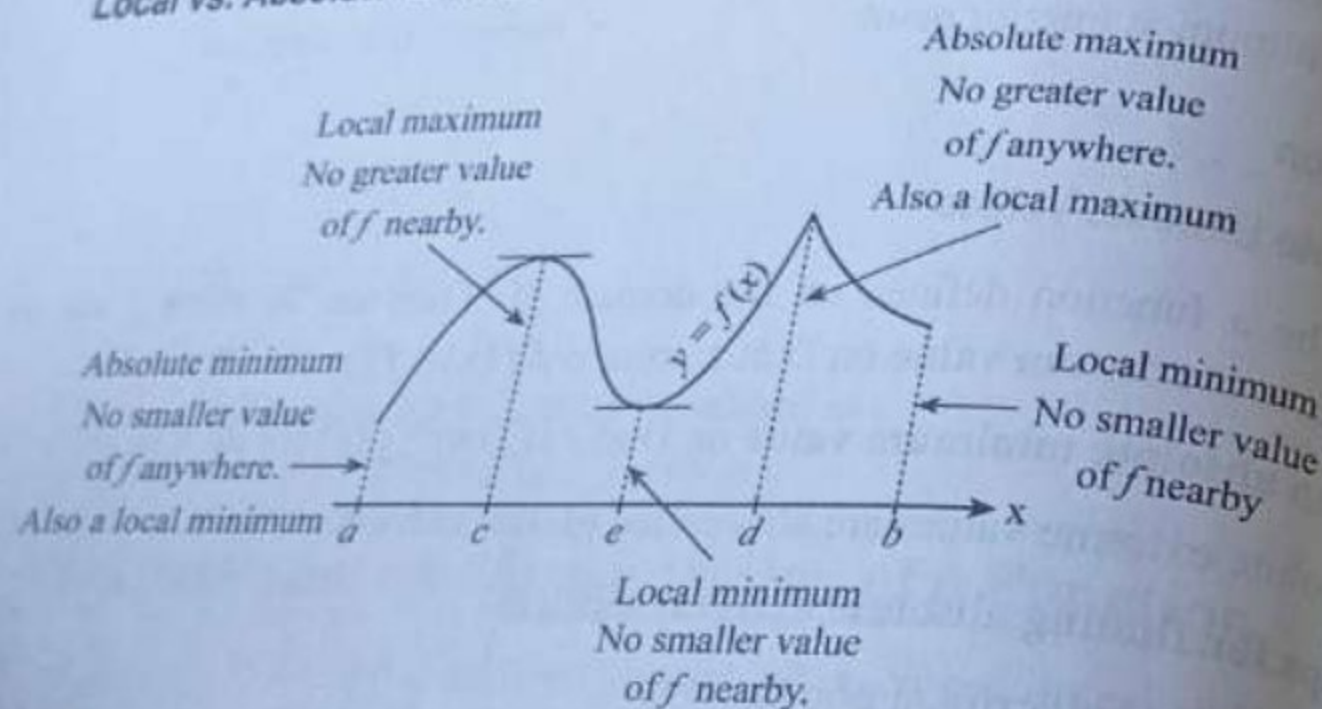
Local Extreme Values

Definition

A function f has a local maximum value at an interior point c of its domain if $f(x) \leq f(c)$ for all x in some interval containing c and a local minimum value at an interior point c of its domain if $f(x) \geq f(c)$ for all x in some open interval containing c .

NOTE Local extreme values are sometimes called relative extreme values.

Local vs. Absolute Extreme



This figure shows a graph with five different extreme points and values.

The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c , then $f'(c) = 0$.

First Derivative Test

- Find the critical points.
- Locate the critical points on the number line.
- Determine the sign of $f'(x)$ for each interval.
- Now let x increase through each critical value $x = c$.

- If $f'(x)$ changes from + to -, then $f(x)$ has a relative (local) maximum value at $x = c$.
- If $f'(x)$ changes from - to +, then $f(x)$ has a relative (local) minimum value at $x = c$.
- If $f'(x)$ does not change sign, then $f(x)$ has neither maximum nor minimum value.

Example 3. Find local extreme values for $f(x) = x^4 - 6x^2 + 8x$.

Solution

Given,

$$f(x) = x^4 - 6x^2 + 8x.$$

Then,

$$f'(x) = 4x^3 - 12x + 8.$$

For the critical points, we have,

$$f'(x) = 0.$$

$$\text{i.e., } 4(x^3 - 3x + 2) = 0$$

$$\text{or, } 4(x-1)^2(x+2) = 0$$

Thus, the critical points are $x = 1$ and $x = -2$.

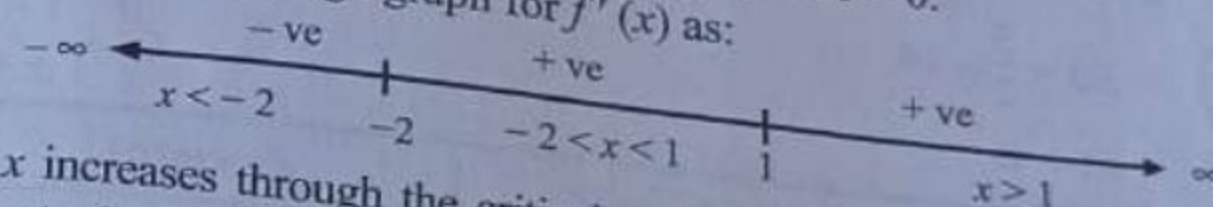
These critical points divide the number line into three intervals:

For $x < -2$: Taking $x = -3$ (test point), $f'(-3) = -64 < 0$.

For $-2 < x < 1$: Taking $x = 0$ (test point), $f'(0) = 4(-1)^2(2) = 8 > 0$.

For $x > 1$: Taking $x = 2$ (test point), $f'(2) = 16 > 0$.

We can draw the sign graph for $f'(x)$ as:



As x increases through the critical value of $x = -2$, $f'(x)$ changes sign from - to +. So $x = -2$ gives us a local minimum at $x = -2$. As x increases through the critical value $x = 1$, $f'(x)$ does not change sign. Thus, there is neither maximum nor minimum at $x = 1$. Hence, the given function has only minimum value and the minimum value is

$$f(-2) = (-2)^4 - 6(-2)^2 + 8(-2) = 16 - 24 - 16 = -24.$$

The Second Derivative Theorem for Local Extreme Values

If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

Producers for finding local extreme values

Step 1: Find $f'(x)$.

Step 2: Solve $f'(x) = 0$ for x . Let c be one such value of x .

Step 3: i. If $f''(x) < 0$ for $x = c$, then $f(x)$ has a maximum value at $x = c$.

- ii. If $f''(x) > 0$ for $x = c$, then $f(x)$ has minimum value at $x = c$.
 iii. $f''(x) = 0$ and $f'''(x) \neq 0$ at $x = c$ then f has neither maximum nor minimum value at $x = c$.

Illustrative Examples

Example 1. Examine whether the function $f(x) = 15x^2 - 14x + 1$ is increasing or decreasing at $x = \frac{2}{5}$ and $x = \frac{5}{2}$.

Solution

Here,

$$f(x) = 15x^2 - 14x + 1$$

$$f'(x) = 30x - 14$$

$$\text{At } x = \frac{2}{5},$$

$$f'\left(\frac{2}{5}\right) = 30 \times \frac{2}{5} - 14$$

$$= -2 < 0$$

$$\therefore f(x) \text{ is decreasing at } x = \frac{2}{5}$$

$$\text{At } x = \frac{5}{2},$$

$$f'\left(\frac{5}{2}\right) = 30 \times \frac{5}{2} - 14$$

$$= 61 > 0$$

$$\therefore f(x) \text{ is increasing at } x = \frac{5}{2}.$$

If $y = f(x)$ represents a certain curve, what do $f'(x) > 0$, $f'(x) < 0$ and $f'(x) = 0$ at a point represent?

Example 2. Find the interval in which the function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is increasing or decreasing. Also, find the point of inflection.

Solution

Here,

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

$$f'(x) = 6x^2 - 30x + 36$$

$$f''(x) = 12x - 30$$

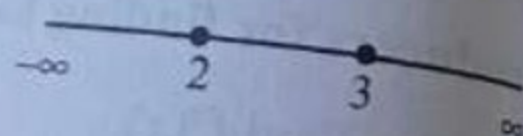
For critical points, we have $f'(x) = 0$

$$\text{or, } 6x^2 - 30x + 36 = 0$$

$$\text{or, } x^2 - 5x + 6 = 0$$

$$\text{or, } (x-2)(x-3) = 0$$

$$\therefore x = 2, 3$$



Intervals	Sign of $f'(x)$	Remarks
$(-\infty, 2)$	+ve	increasing
$(2, 3)$	-ve	decreasing
$(3, \infty)$	+ve	increasing

$\therefore f(x)$ is increasing on $(-\infty, 2) \cup (3, \infty)$ and decreasing on $(2, 3)$.
 For point of inflection, $f''(x) = 0$

$$\text{or, } 12x - 30 = 0$$

$$\text{or, } 12x = 30$$

$$\therefore x = \frac{5}{2}$$

Example 3. Determine where the graph is concave upwards and where it is concave downwards. Also find the point of inflection
 $f(x) = (x^2 - 1)(x^2 - 5)$.

Solution

Here,

$$f(x) = (x^2 - 1)(x^2 - 5)$$

$$= x^4 - 5x^2 - x^2 + 5$$

$$= x^4 - 6x^2 + 5$$

$$f'(x) = 4x^3 - 12x$$

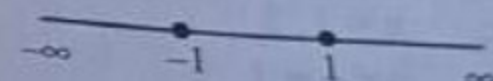
$$f''(x) = 12x^2 - 12$$

For points of inflection $f''(x) = 0$

$$\text{or, } 12x^2 - 12 = 0$$

$$\text{or, } x^2 - 1 = 0$$

$$\therefore x = \pm 1$$



Intervals	Sign of $f''(x)$	Remarks
$(-\infty, -1)$	+ve	concave upwards
$(-1, 1)$	-ve	concave downwards
$(1, \infty)$	+ve	concave upwards

$\therefore$ The graph of $f(x)$ is concave upwards on the interval $(-\infty, -1) \cup (1, \infty)$ and concave downwards on the interval $(-1, 1)$.

Example 4. Find the absolute extrema values of $g(t) = 8t - t^4$ on $[-2, 1]$.

Solution

The function $g(t)$ is differentiable on its entire domain, so the only critical points occur where $g'(t) = 0$.

$$\text{i.e. } 8 - 4t^3 = 0$$

$$\text{or, } t^3 = 2$$

$$\therefore t = \sqrt[3]{2}$$

This point does not belong to $[-2, 1]$.

Therefore, the absolute extreme values occur at the end points.

$$\begin{aligned} \text{Now,} \\ g(-2) &= 8(-2) - (-2)^4 \\ &= -16 - 16 \\ &= -32 \\ g(1) &= 8 \times 1 - 1^4 \\ &= 7. \end{aligned}$$

$\therefore$ Absolute maximum value = 7 at $t = 1$.
Absolute minimum value = -32 at $t = -2$.

Example 5. Find the maximum and minimum values of the function $f(x) = x^3 - 6x^2 + 9x - 2$. Also, find the point of inflection, if any.

Solution

Here,

$$\begin{aligned} f(x) &= x^3 - 6x^2 + 9x - 2 \\ f'(x) &= 3x^2 - 12x + 9 \\ f''(x) &= 6x - 12 \end{aligned}$$

For stationary points, we have $f'(x) = 0$

$$\begin{aligned} \text{or, } 3x^2 - 12x + 9 &= 0 \\ \text{or, } x^2 - 4x + 3 &= 0 \\ \text{or, } (x-1)(x-3) &= 0 \\ \therefore x &= 1, 3 \end{aligned}$$

At $x = 1$

$$\begin{aligned} f''(1) &= 6 \times 1 - 12 \\ &= -6 < 0 \end{aligned}$$

So, $f(x)$ has maximum value at $x = 1$

$$\begin{aligned} \text{Maximum value} &= f(1) \\ &= 1^3 - 6 \times 1^2 + 9 \times 1 - 2 \\ &= 2 \end{aligned}$$

At $x = 3$

$$\begin{aligned} f''(3) &= 6 \times 3 - 12 \\ &= 6 > 0 \end{aligned}$$

So, $f(x)$ has minimum value at $x = 3$

$$\begin{aligned} \text{Minimum value} &= f(3) = 3^3 - 6 \times 3^2 + 9 \times 3 - 2 \\ &= 27 - 54 + 27 - 2 \\ &= -2 \end{aligned}$$

For point of inflection $f''(x) = 0$

$$6x - 12 = 0$$

$$\therefore x = 2.$$

Example 6. Let $f(x)$ be a function whose derivative $f'(x)$ be given by $f'(x) = x(x-1)$.

- What are the critical points of f ?
- On what intervals is f increasing or decreasing?
- At what points, if any, does f assume local maximum and minimum values?

Solution

- Here $f'(x)$ exists for all real values of x .
So, for critical points, we have to solve $f'(x) = 0$.
i.e. $x(x-1) = 0$.
 $\therefore x = 0, 1$.

- The points $x = 0$ and $x = 1$ divide the whole real line into 3 intervals.



Now,

Intervals	Sign of $f'(x)$	Nature of f
$(-\infty, 0)$	+ve	Increasing
$(0, 1)$	-ve	Decreasing
$(1, \infty)$	+ve	Increasing

- $f(x)$ is decreasing on $(0, 1)$ and increasing on $(-\infty, 0) \cup (1, \infty)$.

$$\begin{aligned} \text{Here,} \\ f'(x) &= x(x-1) \\ &= x^2 - x \end{aligned}$$

$$f''(x) = 2x - 1$$

The critical points are $x = 0$ and $x = 1$.

At $x = 0$,

$$\begin{aligned} f''(x) &= 2 \times 0 - 1 \\ &= -1 < 0. \end{aligned}$$

So, f has local maximum value at $x = 0$.

At $x = 1$,

$$\begin{aligned} f''(x) &= 2 \times 1 - 1 \\ &= 1 > 0. \end{aligned}$$

So, f has local minimum value at $x = 1$.

Example 7. Show that the rectangle of largest possible area for a given perimeter is a square.

Solution

Let x and y be the length and breadth, A be the area and P be the given perimeter of the given rectangle.

We have,

$$P = 2(x + y)$$

2. Find the intervals in which the following functions are increasing or decreasing.

a. $f(x) = x^2 - 2x$

b. $f(x) = -x^2 - 3x + 3$

c. $f(x) = x^3 - 3x^2 - 9x$

d. $f(x) = -x^3 + 12x + 5, -3 \leq x \leq 3$

3. Find the absolute maximum and minimum values of each function on the given interval.

a. $f(x) = \frac{2}{3}x - 5, -2 \leq x \leq 3$

b. $f(x) = x^2 - 1, -1 \leq x \leq 2$

c. $f(x) = x^3 - 3x^2 + 5$ on $[-2, 2]$

d. $h(x) = x^{\frac{2}{3}}$ on $[-2, 3]$

4. Find the local maximum and minimum values and point of inflection of the following functions.

a. $f(x) = 3x^2 - 6x + 4$

b. $f(x) = x^3 - 3x + 1$

c. $f(x) = x^3 - 3x^2 - 9x + 27$

d. $f(x) = 2x^3 - 15x^2 + 36x + 5$

e. $y = 4x^3 - 6x^2 - 9x + 1$ on the interval $(-1, 2)$.

f. $y = x + \frac{25}{x}$

5. Show that the function $x^3 + 6x + 30$ has neither a maximum value nor a minimum value.

6. Find the interval in which the given functions are concave upwards and downwards.

a. $f(x) = x^2 - 3x + 1$

b. $f(x) = x^3 - 3x^2 + 5$

c. $f(x) = x^4 - 8x^3 + 18x^2 - 24$

7. A man who has 144 m of fencing material wishes to enclose a rectangular garden. Find the maximum area he can enclose.

8. A gardener having 120 m of fencing wishes to enclose a rectangular plot of land and also to erect a fence across the land parallel to two of the sides. Find the maximum area he can enclose.

9. Using derivatives, find two numbers whose sum is 20 and sum of whose squares is minimum.

10. A window is in the form of a rectangle surmounted by a semi circle. If the total perimeter is 9 m, find the radius of the semicircle for the greatest window area.

11. A closed cylindrical can is to be made so that its volume is 26 cm^3 . Find its radius and height if the surface is to be a minimum.

Very Short Answer Questions

- Find the rate of change of $y = x^2 - 3x + 5$ with respect to x .
- What are the critical points of the function $f(x) = 2x^3 - 6x^2 + 5$?
- Write the condition for the function $f(x)$ to be increasing on a domain D .
- What is the absolute maximum value of $f(x) = x^2$ on $[-2, 1]$?
- Write the conditions for the function $y = f(x)$ to have local minimum value?
- What are the points of inflection of the function $y = x^4 - 4x^3 + 10$?

Answers

- b. Increasing at $x = 1$; Decreasing at $x = \frac{1}{4}$
- a. Increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$
b. Increasing on $(-\infty, -\frac{3}{2})$ and decreasing on $(-\frac{3}{2}, \infty)$
c. Increasing on $(-\infty, -1) \cup (3, \infty)$ and decreasing on $(-1, 3)$
d. Increasing on $(-2, 2)$ and decreasing on $[-3, -2) \cup (2, 3]$
- a. Absolute max. value = -3 at $x = 3$, Absolute min. value = $-\frac{19}{3}$ at $x = -2$
b. Absolute max. value = 3 at $x = 2$, Absolute min. value = -1 at $x = 0$
c. Absolute max. value = 5 at $x = 0$, Absolute min. value = -15 at $x = -2$
d. Absolute max. value = $\sqrt[3]{9}$ at $x = 3$, Absolute min. value = 0 at $x = 0$
- a. Min. value = 1 at $x = 1$, no point of inflection
b. Max. value = 3 at $x = -1$, Min value = -1 at $x = 1$; point of inflection is at $x = 0$
c. Max. value = 32 at $x = -1$; Min value = 0 at $x = 3$; Point of inflection is at $x = 1$
d. Max. value = 33 at $x = 2$; Min value = 32 at $x = 3$; Point of inflection is at $x = \frac{5}{2}$
e. Max. value = $\frac{7}{2}$ at $x = -\frac{1}{2}$; Min value = $-\frac{25}{2}$ at $x = \frac{3}{2}$; Point of inflection is at $x = \frac{1}{2}$
f. Max. value = -10 at $x = 5$; Min. value = 10 at $x = -5$, no point of inflection
- a. Concave upwards on $(-\infty, \infty)$
b. Concave upwards on $(1, \infty)$ and downwards on $(-\infty, 1)$
c. Concave upwards on $(-\infty, 1) \cup (3, \infty)$; Concave downwards on $(1, 3)$
- 1296 m^2
- 600 m^2
- $10, 10$
- $\frac{9}{4 + \pi} \text{ m}$
- $r = \left(\frac{13}{\pi}\right)^{\frac{1}{3}}, h = 2\left(\frac{13}{\pi}\right)^{\frac{1}{3}}$

Very Short Answer Questions

- $2x - 3$
- $0, 2$
- 4
- 4
- $0, 2$

3.6 Tangents and Normals

One of the problems which developed differential calculus was that of finding the slope of the tangent to a curve. We first define a tangent and normal to any curve at a given point on it.

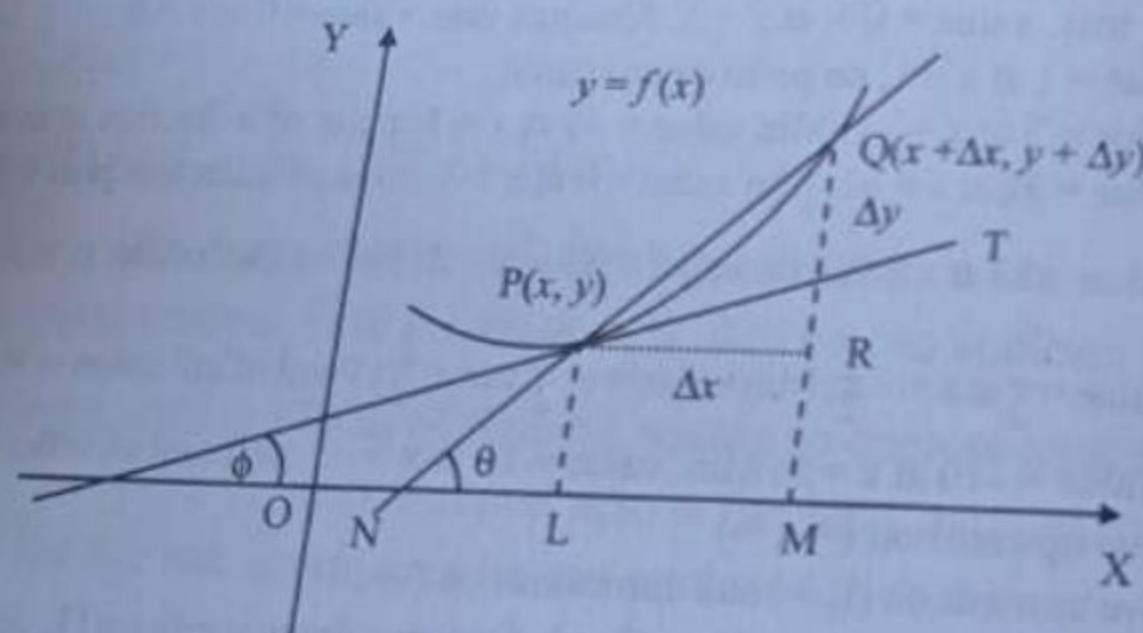
Definition

Let P be any fixed point on a given curve and Q be a variable neighboring point on the curve as shown in the figure. Join P and Q . Let Q move along the curve and approaches P ; the secant line PQ will rotate about the point P , and PQ may approach a limiting position PT . The limiting position PT of PQ as Q approaches P along the curve is called the **tangent line** to the curve at the point P . The line which is perpendicular to the tangent at the point P is called a **normal** at the point P .

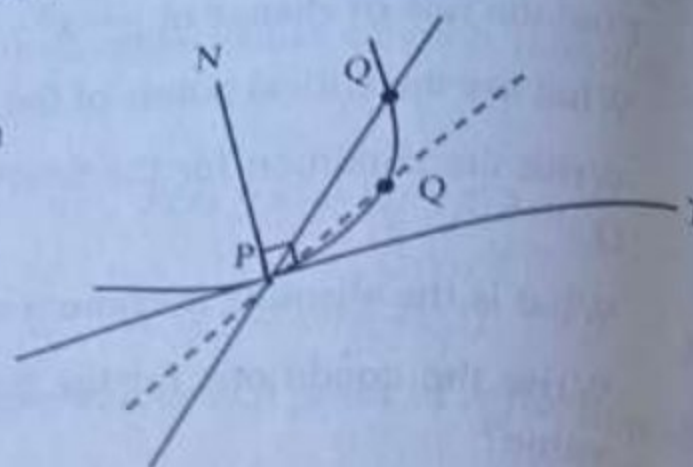
Let $y = f(x)$ be a curve and $P(x, y)$ be a fixed point and $Q(x + \Delta x, y + \Delta y)$ be a neighboring point on the curve. From P and Q , draw PL and QM perpendiculars to the x -axis. Also draw perpendicular PR on QM . Now, $QR = \Delta y$ and $PR = LM = \Delta x$.

$$\text{Slope of the secant } PQ = \tan \theta = \frac{QR}{PR} = \frac{\Delta y}{\Delta x}.$$

The ratio $\frac{\Delta y}{\Delta x}$ is called difference quotient, where $\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$.



Now, when $Q \rightarrow P$, $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$, $\theta \rightarrow \phi$, then PT will be tangent at P . Thus, the slope of tangent PT is $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \tan \phi$, provided that the limit exists.



By the definition of derivative, $f'(x) = \tan \phi$ is the slope of the tangent at P . Hence, the slope of the tangent to a curve $y = f(x)$ at a point (x, y) is given by the value of the derivative of the function $f(x)$ with respect to x at that point.

NOTE

1. The derivative of $f(x)$ at $x = a$ is written as

$$f'(a) = \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

2. The slope of the tangent to a curve is sometimes called the slope of the curve.

3. If the tangent is parallel to x axis, then $\phi = 0^\circ$, so $\frac{dy}{dx} = 0$.

4. If the tangent is perpendicular to x axis, then $\phi = 90^\circ$.

$$\therefore \frac{dy}{dx} = \tan \phi = \tan 90^\circ = \infty, \text{ i.e., } \frac{dx}{dy} = 0.$$

3.6.1 Equation of Tangents and Normals

Let $y = f(x)$ be a given curve and $P(x_1, y_1)$ be a point on the curve.

The equation of any straight line passing through the point $P(x_1, y_1)$ and having slope m is

$$y - y_1 = m(x - x_1)$$

The line becomes the tangent at $P(x_1, y_1)$ if

$$m = \left(\frac{dy}{dx} \right) \text{ at } P(x_1, y_1)$$

Hence, the equation of tangent to the curve

$$y = f(x) \text{ at } P(x_1, y_1) \text{ is}$$

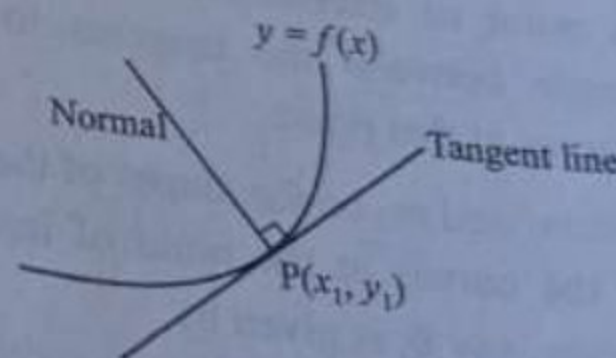
$$y - y_1 = \left(\frac{dy}{dx} \right)_{(x, y) = (x_1, y_1)} (x - x_1) \quad \dots (i)$$

The normal to the curve at $P(x_1, y_1)$ is perpendicular to the tangent to the curve at $P(x_1, y_1)$

$\therefore$ Slope of normal to the curve at $P(x_1, y_1)$

$$= - \frac{1}{\text{Slope of tangent of the curve at } P(x_1, y_1)}$$

$$= - \frac{1}{\left(\frac{dy}{dx} \right)_{(x, y) = (x_1, y_1)}} = - \left(\frac{dx}{dy} \right)_{(x, y) = (x_1, y_1)}$$



Equation of normal to the curve

$$y = f(x) \text{ at } P(x_1, y_1) \text{ is}$$

$$y - y_1 = -\left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

NOTE

1. If a tangent to the curve $y = f(x)$ at P makes angle θ with positive x -axis, then $\frac{dy}{dx}$ = slope of tangent at $P = \tan \theta$
2. The tangent to $y = f(x)$ at P is horizontal if and only if $\frac{dy}{dx} = 0$ at P .
3. The tangent to $y = f(x)$ at P is vertical if and only if $\frac{dx}{dy} = 0$ at P .

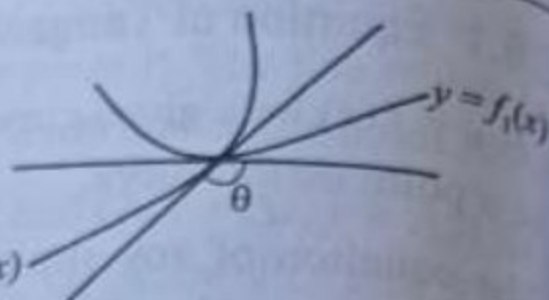
3.7 Angle between Two Curves

The angle of intersection of two curves at a point of intersection is defined as the angle between the tangents to the two curves at that point.

Let m_1 and m_2 be the slopes of the tangents to the curves at the point of intersection then the angle of intersection of curves, say θ , is given by

$$\tan \theta = \pm \frac{m_1 - m_2}{m_1 + m_2}$$

If $m_1 m_2 = -1$ then the two curves cut orthogonally.



Illustrative Examples

Example 1. If $f(x) = x^2 + 2x + 1$, find Δy and dy .

Solution

Here,

$$y = x^2 + 2x + 1$$

Then,

$$y + \Delta y = (x + \Delta x)^2 + 2(x + \Delta x) + 1$$

$$\text{or, } \Delta y + x^2 + 2x\Delta x + (\Delta x)^2 + 2x + 2\Delta x + 1 = x^2 + 2x + 1$$

$$\text{or, } \Delta y = 2x\Delta x + (\Delta x)^2 + 2\Delta x$$

$$\therefore \Delta y = (2x + 2 + \Delta x) \Delta x$$

and

$$dy = f'(x) dx$$

$$\therefore dy = (2x + 2) dx.$$

Example 2. A circular copper plate is heated so that its radius increases from 5 cm to 5.06 cm. Find the approximate increase in area.

Solution

Let ' r ' be the radius and ' A ' be the area of the circular plate. Given, $r = 5$ cm, $\Delta r = dr = 5.06 - 5 = 0.06$

We have,

$$A = \pi r^2$$

$$dA = 2\pi r dr = 2\pi \times 5 \times 0.06 = 0.6\pi \text{ cm}^2$$

$$\therefore \text{Approximate increase in area} = 0.6\pi \text{ cm}^2.$$

Example 3. Find the slope and inclination with x -axis of the tangent of the curve $2y = 2 - x^2$ at $x = 1$.

Solution

Given curve is $2y = 2 - x^2$

Differentiating both sides with respect to ' x '

$$2 \frac{dy}{dx} = 0 - 2x$$

$$\frac{dy}{dx} = -x$$

$$\text{At } x = 1, \frac{dy}{dx} = -1$$

$$\text{i.e. Slope } (m) = -1$$

If θ be the inclination of the tangent with x -axis, then $\tan \theta = -1 = \tan \frac{3\pi}{4}$

$$\therefore \theta = \frac{3\pi}{4}.$$

Example 4. Find the slope of the parabola $y = x^2$ at the point $p(2, 4)$. Write an equation for the tangent to the parabola at this point.

Solution

Given $y = x^2$

$$\frac{dy}{dx} = 2x$$

Now,

$$\frac{dy}{dx} \text{ at } (x, y) = (2, 4) = 2 \times 2 = 4.$$

$\therefore$ Slope of the given parabola at $(2, 4)$ is 4.

The equation of tangent to the parabola at $p(2, 4)$ is

$$y - 4 = 4(x - 2)$$

$$\text{or, } y - 4 = 4x - 8$$

$$\text{or, } 0 = 4x - y - 8 + 4$$

$$\therefore 4x - y - 4 = 0.$$

Example 5. a. Find the slope of the curve $y = \frac{1}{x}$ at $x = a$.
 b. Where does the slope equal to $-\frac{1}{4}$?

Solution

Given $y = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-1}) = -1 \cdot x^{-1-1}$$

$$= -\frac{1}{x^2}$$

a. Slope of curve at $x = a$ is $\left(\frac{dy}{dx}\right)_{\text{at } x=a}$

$$= \left(-\frac{1}{x^2}\right)_{\text{at } x=a}$$

$$= -\frac{1}{a^2}$$

b. Slope of curve $= -\frac{1}{4}$

or, $-\frac{1}{x^2} = -\frac{1}{4}$

or, $x^2 = 4$

or, $x^2 = (\pm 2)^2$

$\therefore x = \pm 2$

When $x = 2, y = \frac{1}{2}$

When $x = -2, y = -\frac{1}{2}$

Thus, the curve has slope $-\frac{1}{4}$ at the two points $\left(2, \frac{1}{2}\right)$ and $\left(-2, -\frac{1}{2}\right)$.

Example 6. Find the equations of the tangent and normal to the curve $y = 2x^2 - 5x + 2$ at $(1, -1)$.

Solution

Given curve is $y = 2x^2 - 5x + 2$

$$\frac{dy}{dx} = 4x - 5$$

Slope of tangent at $(1, -1) = \left(\frac{dy}{dx}\right)_{\text{at } (x, y) = (1, -1)}$

$$= 4 \times 1 - 5$$

$$= -1.$$

Slope of normal at $(1, -1) = -\frac{1}{\text{slope of tangent}} = 1$

The equation of tangent at $(x_1, y_1) = (1, -1)$ is
 $y - y_1 = m(x - x_1)$ where $m = \text{slope of tangent}$

or, $y + 1 = -1(x - 1)$

or, $y + 1 = -x + 1$

or, $x + y = 0.$

$\therefore$ The equation of normal at $(x_1, y_1) = (1, -1)$ is
 $y - y_1 = m(x - x_1)$, where $m = \text{slope of normal}$

or, $y + 1 = 1(x - 1)$

or, $y + 1 = x - 1$

or, $0 = x - y - 1 - 1$

$\therefore x - y = 2.$

Example 7. Find where the tangent is parallel to the x-axis for the curve $y = x^3 - 3x^2 - 9x + 15$.

Solution

Given,

$$y = x^3 - 3x^2 - 9x + 15$$

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

For the tangent to be parallel to the x-axis, we have

$$\frac{dy}{dx} = 0$$

i.e. $3x^2 - 6x - 9 = 0$

or, $x^2 - 2x - 3 = 0$

or, $(x - 3)(x + 1) = 0$

$\therefore x = -1, 3$

When $x = -1,$

$$y = (-1)^3 - 3 \times (-1)^2 - 9 \times (-1) + 15$$

$$= -1 - 3 + 9 + 15$$

$$= 20$$

When $x = 3,$

$$y = 3^3 - 3 \times 3^2 - 9 \times 3 + 15$$

$$= -12$$

$\therefore$ Required points are $(-1, 20)$ and $(3, -12)$.

Example 8. Find the points on the curve $x^2 + y^2 = 36$ at which the tangents are parallel to the y-axis.

Solution

Given, curve is $x^2 + y^2 = 36$... (i)

Differentiating both sides w.r.to x

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(36)$$

$$\text{or, } 2x + 2y \frac{dy}{dx} = 0$$

$$\text{or, } \frac{dy}{dx} = -\frac{x}{y}$$

For the tangent parallel to y-axis, we have,

$$\frac{dy}{dx} = 0$$

$$\text{or, } -\frac{x}{y} = 0$$

$$\therefore y = 0$$

Putting the value of y in (i), we get,

$$x^2 = 36$$

$$\text{or, } x = \pm 6.$$

Hence, the required points are (6, 0) and (-6, 0).

Example 9. Find the angle of intersection between the curves $y = x^2$ and $6y = 7 - x^2$ at (1, 1).

Solution

Given curves are

$$y = x^2 \quad \dots(i)$$

and

$$6y = 7 - x^2 \quad \dots(ii)$$

Differentiating (i) with respect to 'x'

$$\frac{dy}{dx} = 2x$$

At (1, 1);

$$\frac{dy}{dx} = 2 \times 1 = 2$$

$$\text{i.e. } m_1 = 2$$

Again, differentiating (ii) both sides with respect to 'x'

$$6 \frac{dy}{dx} = -3x^2$$

$$\text{or, } \frac{dy}{dx} = -\frac{x^2}{2}$$

At (1, 1);

$$\frac{dy}{dx} = -\frac{1}{2}$$

$$\text{i.e. } m_2 = -\frac{1}{2}$$

If θ be the angle of intersection, then

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{2 + \frac{1}{2}}{1 + 2 \cdot \left(-\frac{1}{2}\right)}$$

$$= \left(\frac{5}{0}\right)$$

$$= \infty$$

$$= \tan 90^\circ$$

$$= 90^\circ.$$

Exercise 3.2

1. Find the slope and inclination with x-axis of the tangent of the following curves

a. $3y = x^3 + 1$ at $x = 1$

b. $y = -x^4 - 3x$ at $(-1, 2)$

c. $x^2 + y^2 = 25$ at $(4, -3)$

2. At what angle does the curve $y(1+x) = x$ cut x-axis?

3. Find the equations of the tangents and normals to the following curves.

a. $y = 2x^3 - 5x^2 + 8$ at $(2, 4)$

b. $x^2 + y^2 = 25$ at $(3, 4)$

4. Find the points on the following curves where the tangents are parallel to the x-axis

a. $y = x^2 + 4x + 1$

b. $y = x^3 - 2x^2 + 1$

5. Find the points on the curve $x^2 + y^2 = 25$ at which the tangents are parallel to the (a) x-axis (b) y-axis.

6. a. Find the equation of the tangent to the curve $y = 2x^2 - 3x + 1$ which is parallel to the line $x - y + 5 = 0$.

b. Find the equation of the tangent to the curve $y = 3x^2 - 3x + 5$ which is perpendicular to the line $x + 3y + 5 = 0$.

7. Find the angle of intersection of the following curves.

a. $y^2 = x^3$ and $y = 2x$ at $(0, 0)$

b. $xy = 6$ and $x^2y = 12$

Very Short Answer Questions

- Write the slope of tangent to the curve $y = f(x)$ at the point (x_1, y_1) .
- Write the equation of normal to the curve $y = f(x)$ at $P(x_1, y_1)$.
- Find the slope of tangent to the curve $y = x^2 + 5x$ at $(1, 2)$.
- Find x -coordinates of the point on the curve $y = x^2 + 4x$ where the tangent is parallel to x -axis.
- Find the slope of normal to the curve $x^2 + y^2 = 25$ at $(3, 4)$.

Answers

- a. $1, \frac{\pi}{4}$ b. $1, \frac{\pi}{4}$ c. $\frac{4}{3}, \tan^{-1}\left(\frac{4}{3}\right)$
- a. $\frac{\pi}{4}$ b. $3x + 4y = 25, 4x - 3y = 0$
- a. $4x - y = 4, x + 4y = 18$ b. $(0, 1)$ and $\left(\frac{4}{3}, -\frac{5}{27}\right)$
- a. $(-2, -3)$ b. $(\pm 5, 0)$
- a. $(0, \pm 5)$ b. $3x - y + 2 = 0$
- a. $x - y - 1 = 0$ b. $\tan^{-1}\left(\frac{3}{11}\right)$
- a. $\tan^{-1}(2)$

Very Short Answer Questions

- 7
- $(-2, -4)$
- $\frac{4}{3}$

3.8 Indeterminate Forms and L Hospital's Rule

If functions $f(x)$ and $g(x)$ are both zero at $x = a$ then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ cannot be found by substituting $x = a$. The substitution given $\frac{0}{0}$ which is meaningless, known as indeterminate form.

Other indeterminate forms are, $\frac{\infty}{\infty}, \infty - \infty, 0^0, \infty^0, 1^\infty$.

Here we discuss the idea of finding limiting values of the functions which take the form $\frac{0}{0}$ and $\frac{\infty}{\infty}$ at $x = a$.

L' Hospital's rule help us to get success with derivatives to evaluate limits that lead to indeterminate forms.

L Hospital's Rule

If $f(x)$ and $g(x)$ and their derivatives $f'(x)$ and $g'(x)$ are continuous at $x = a$ and $f(a) = g(a) = 0$ then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$ provided that $g'(a) \neq 0$.

If $f''(a) = g''(a) = 0$ then above rule can be extended as

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow a} \frac{f''(x)}{g''(x)} = \frac{f''(a)}{g''(a)}$$

provided that $f''(x)$ and $g''(x)$ are continuous at $x = a$ and $g''(a) \neq 0$.

The above rule can further be used when $f''(a) = g''(a) = 0$

Example: It tooks a lot of work to evaluate $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ in previous section. But it is easy with the help of this rule.

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sin x}{x} \left[\frac{0}{0} \text{ form} \right] \\ &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)} \\ &= \lim_{x \rightarrow 0} \frac{\cos x}{1}, \text{ Not } \frac{0}{0}, \text{ limit is found.} \\ &= \frac{\cos 0}{1} = \frac{1}{1} = 1. \end{aligned}$$

Illustrative Examples

Example 1. Using L Hospital's rule, evaluate $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$.

Solution

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} \text{ still } \frac{0}{0} \quad \text{apply the rule again}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{6x} \text{ still } \frac{0}{0}, \quad \text{apply the rule again}$$

$$= \lim_{x \rightarrow 0} \frac{\cos}{6}$$

$$= \frac{1}{6} \text{ Not } \frac{0}{0}.$$

limit is found.

Example 2. Use L Hospital's rule to evaluate $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$.

Solution

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2-4} \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 2} \frac{1}{2x}$$

$$= \frac{1}{2} \times 2$$

$$= \frac{1}{4}.$$

Example 3. Use L Hospital's rule to evaluate $\lim_{x \rightarrow -3} \frac{x^3-4x+15}{x^2-x-12}$.

Solution

$$\lim_{x \rightarrow -3} \frac{x^3-4x+15}{x^2-x-12} \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow -3} \frac{3x^2-4}{2x-1}$$

$$= \frac{3(-3)^2-4}{2(-3)-1}$$

$$= -\frac{23}{7}.$$

Example 4. Using L Hospital's rule, evaluate: $\lim_{x \rightarrow 0} \frac{x^2-\sin^2 x}{x^2}$.

Solution

$$\lim_{x \rightarrow 0} \frac{x^2-\sin^2 x}{x^2} \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{2x-2\sin x \cos x}{2x} \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{2x-\sin 2x}{2x}$$

$$= \lim_{x \rightarrow 0} \frac{2-2\cos 2x}{2}$$

$$= \frac{2-2\cos 0}{2} = \frac{2-2 \times 1}{2}$$

$$= \frac{2-2}{2} = \frac{0}{2} = 0.$$

Example 5. Using L Hospital's rule, evaluate: $\lim_{x \rightarrow 0} \frac{e^x-x-1}{x^2}$.

Solution

$$\lim_{x \rightarrow 0} \frac{e^x-x-1}{x^2} \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{e^x-1}{2x} \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{e^x}{2}$$

$$= \frac{e^0}{2}$$

$$= \frac{1}{2}.$$

Example 6. Using L Hospital's rule, evaluate: $\lim_{x \rightarrow 0} \frac{xe^x - \log(1+x)}{x^2}$.

Solution

$$\lim_{x \rightarrow 0} \frac{xe^x - \log(1+x)}{x^2} \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{xe^x + e^x - \frac{1}{1+x}}{2x} \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{xe^x + e^x + e^x + \frac{1}{(x+1)^2}}{2}$$

$$= \frac{0+1+1+1}{2}$$

$$= \frac{3}{2}.$$

Example 7. Using L Hospital's rule, evaluate: $\lim_{x \rightarrow \infty} \frac{3x^2+2x-5}{2x^2+7x+8}$.

Solution

$$\lim_{x \rightarrow \infty} \frac{3x^2+2x-5}{2x^2+7x+8} \left[\frac{\infty}{\infty} \text{ form} \right]$$

$$= \lim_{x \rightarrow \infty} \frac{6x+2}{4x+7} \left[\frac{\infty}{\infty} \text{ form} \right]$$

$$= \lim_{x \rightarrow \infty} \frac{6}{4}$$

$$= \frac{3}{2}.$$

Example 8. Evaluate, using L'Hospital's rule: $\lim_{x \rightarrow 0} \frac{\ln(\tan x)}{\ln x}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\ln(\tan x)}{\ln x} & \left[\frac{\infty}{\infty} \text{ form} \right] \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{\tan x} \cdot \sec^2 x}{\frac{1}{x}} \\ &= \lim_{x \rightarrow 0} \frac{x \sec^2 x}{\tan x} \left[\frac{0}{0} \text{ form} \right] \\ &= \lim_{x \rightarrow 0} \frac{x \cdot 2 \sec x \cdot \sec x \tan x + \sec^2 x}{\sec^2 x} \\ &= \frac{0+1}{1} \\ &= 1. \end{aligned}$$

Exercise 3.3

1. Evaluate the following using L'Hospital's rule.

a. $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$

c. $\lim_{x \rightarrow -5} \frac{x^2 - 25}{x + 5}$

e. $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$

g. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2}$

i. $\lim_{x \rightarrow 0} \frac{\tan x - x}{x - \sin x}$

k. $\lim_{x \rightarrow 0} \frac{(e^x - 1) \tan x}{x^2}$

b. $\lim_{x \rightarrow 3} \frac{x^3 - 27}{x^2 - 9}$

d. $\lim_{x \rightarrow 2} \frac{x^2 - 5x + 6}{x^2 - x - 2}$

f. $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2}$

h. $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{x^2}$

j. $\lim_{x \rightarrow 0} \frac{x - \sin x \cos x}{x^3}$

l. $\lim_{t \rightarrow 0} \frac{t \sin t}{1 - \cos t}$

2. Evaluate the following limits using L'Hospital's rule.

a. $\lim_{x \rightarrow \infty} \frac{5x^2 - 3x}{7x^2 + 1}$

b. $\lim_{x \rightarrow \infty} \frac{2x^2 + 3x}{1 + 5x^2}$

c. $\lim_{x \rightarrow \infty} \frac{x^5}{e^x}$

d. $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan x}{\tan 3x}$

Very Short Answer Questions

1. Define L'Hospital's rule.

2. Write the value of $\lim_{x \rightarrow 0} \frac{x}{\sin x}$.

3. Is $\lim_{x \rightarrow 2} \frac{x-2}{x^2-2} = \frac{0}{2} = 0$ correct? Justify.

4. Evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2}$.

5. $\lim_{x \rightarrow \infty} \frac{3x^2 + 5x + 1}{4x^2 + 6x + 5}$ using L'Hospital's rule.

Answers

- | | | | |
|---------------------|-------------------|--------|-------------------|
| 1. a. 8 | b. $\frac{9}{2}$ | c. -10 | d. $-\frac{1}{3}$ |
| e. $\frac{1}{2}$ | f. $-\frac{1}{8}$ | g. 0 | h. $\frac{1}{2}$ |
| i. 2 | j. $\frac{2}{3}$ | k. 1 | l. 2 |
| 2. a. $\frac{5}{7}$ | b. $\frac{2}{5}$ | c. 0 | d. 3 |
| 4. 0 | 5. $\frac{3}{4}$ | | |

Very Short Answer Questions

2. 1

3.9 Rolle's Theorem

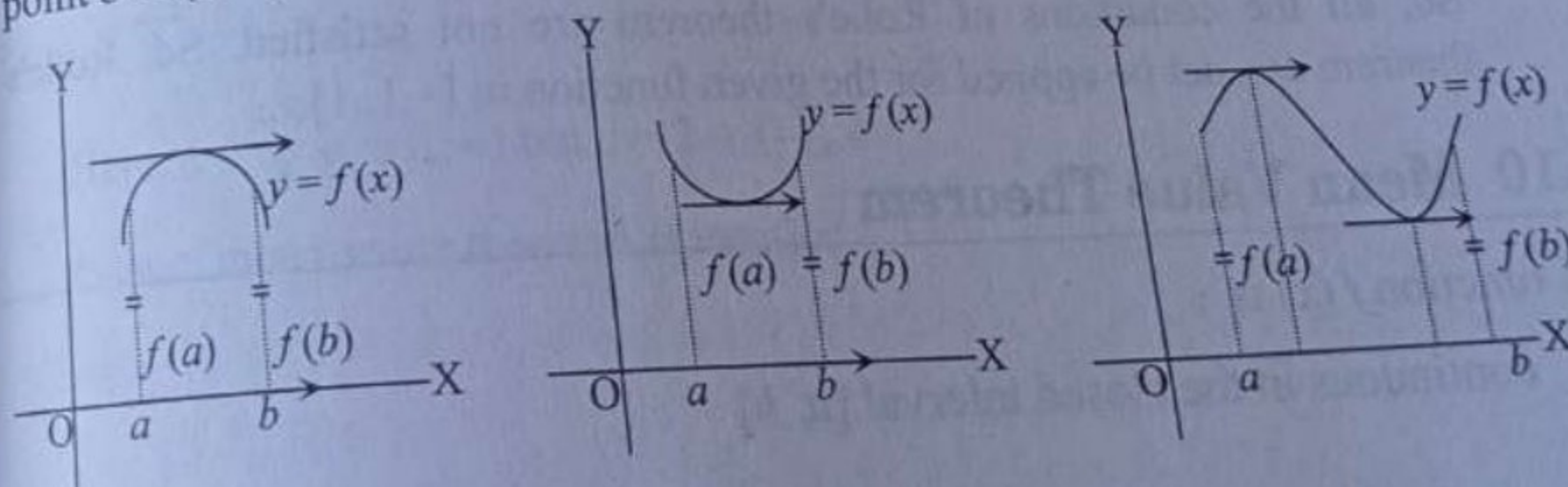
If a function $f(x)$ is

- continuous in $[a, b]$
- differentiable in (a, b)
- $f(a) = f(b)$

then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Geometrical interpretation

If all the conditions of Rolle's theorem are satisfied then there is at least one point $c \in (a, b)$ where the tangent is parallel to x-axis.



Example 1. Verify Rolle's Theorem for the function $f(x) = x(x-3)^2$ for $x \in [0, 3]$.

Solution

Here,

$$\begin{aligned} f(x) &= x(x-3)^2, x \in [0, 3] \\ &= x(x^2 - 6x + 9) \\ &= x^3 - 6x^2 + 9x \end{aligned}$$

Since $f(x)$ is a polynomial function, so it is continuous in $[0, 3]$.

Again, $f'(x) = 3x^2 - 12x + 9$ which exists for all $x \in (0, 3)$.

So, $f(x)$ is differentiable in $(0, 3)$.

And

$$f(0) = 0^3 - 6 \times 0^2 + 9 \times 0 = 0$$

$$f(3) = 3^3 - 6 \times 3^2 + 9 \times 3 = 0$$

$$\therefore f(0) = f(3).$$

All the conditions of Rolle's theorem are satisfied. So there exists at least one

$$c \in (0, 3) \text{ such that } f'(c) = 0$$

$$\Rightarrow 3c^2 - 12c + 9 = 0$$

$$\Rightarrow c^2 - 4c + 3 = 0$$

$$\Rightarrow (c-1)(c-3) = 0$$

$$\Rightarrow c = 1, 3$$

Here, $c = 1 \in (0, 3)$ but $c = 3 \notin (0, 3)$.

Hence Rolle's theorem is verified.

Failure of Rolle's Theorem

If anyone of the three conditions of Rolle's theorem are not satisfied then Rolle's theorem will not be true.

Example 2. Show that Rolle's theorem can not be applied for the function $f(x) = x^{\frac{2}{3}}$ in $[-1, 1]$.

Solution

For all $x \in [-1, 1]$, $f(x)$ has a definite value, so $f(x)$ is continuous in $[-1, 1]$.

Here, $f'(x) = \frac{2}{3x^{\frac{1}{3}}}$ which does not exist for $x = 0 \in (-1, 1)$.

So, $f(x)$ is not differentiable at $x = 0$ in $(-1, 1)$.

So, all the conditions of Rolle's theorem are not satisfied. So, Rolle's theorem can not be applied for the given function in $[-1, 1]$.

3.10 Mean Value Theorem

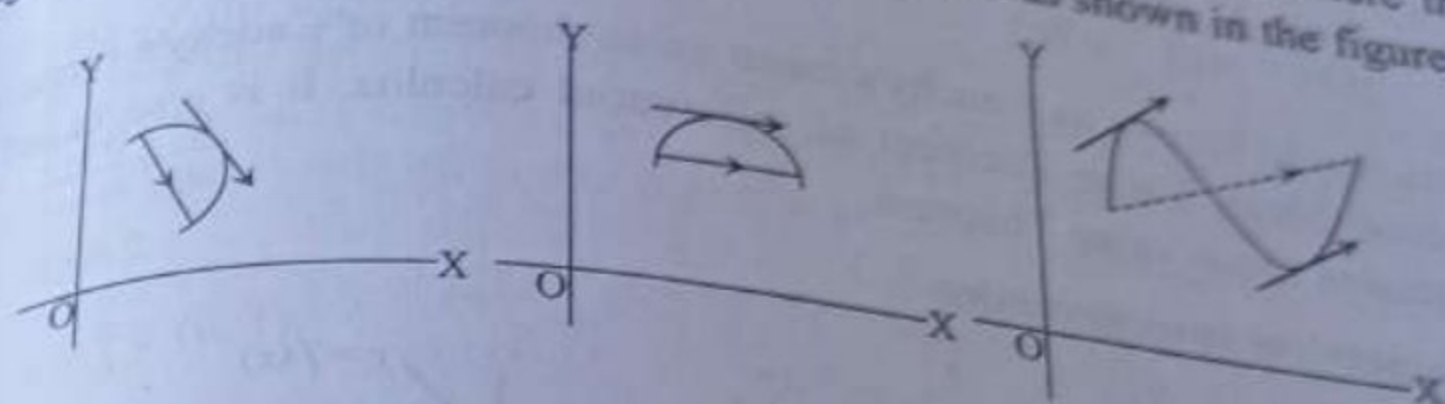
If a function $f(x)$ is

- continuous in the closed interval $[a, b]$

differentiable in the open interval (a, b) then there exists at least one $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Geometrical interpretation

Lagrange's mean value theorem says that in a continuous curve, in which tangent can be drawn at every point, there is at least one point where the tangent is parallel to the secant joining the end points as shown in the figure.



Example 3. Verify mean value theorem for the function $f(x) = x^3 + x^2 - 6x$ in $[-1, 4]$.

Solution

Here, $f(x) = x^3 + x^2 - 6x$

Since $f(x)$ is a polynomial function, so it is continuous in $[-1, 4]$.

Again, $f'(x) = 3x^2 + 2x - 6$ which exists for all $x \in (-1, 4)$.

$\therefore f(x)$ is differentiable in $(-1, 4)$.

So both conditions of mean value theorem are satisfied. Hence there exists at least one c in $(-1, 4)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow 3c^2 + 2c - 6 = \frac{56 - 6}{4 - (-1)}$$

$$\Rightarrow 3c^2 + 2c - 6 = 10$$

$$\Rightarrow 3c^2 + 2c - 16 = 0$$

$$\Rightarrow 3c^2 + 8c - 6c - 16 = 0$$

$$\Rightarrow c(3c + 8) - 2(3c + 8) = 0$$

$$\Rightarrow (3c + 8)(c - 2) = 0$$

$$\Rightarrow c = \frac{-8}{3}, 2$$

Here, $c = \frac{-8}{3} \notin (-1, 4)$ but $c = 2 \in (-1, 4)$

Hence, mean value theorem is verified.

3.11 Generalized Mean Value Theorem

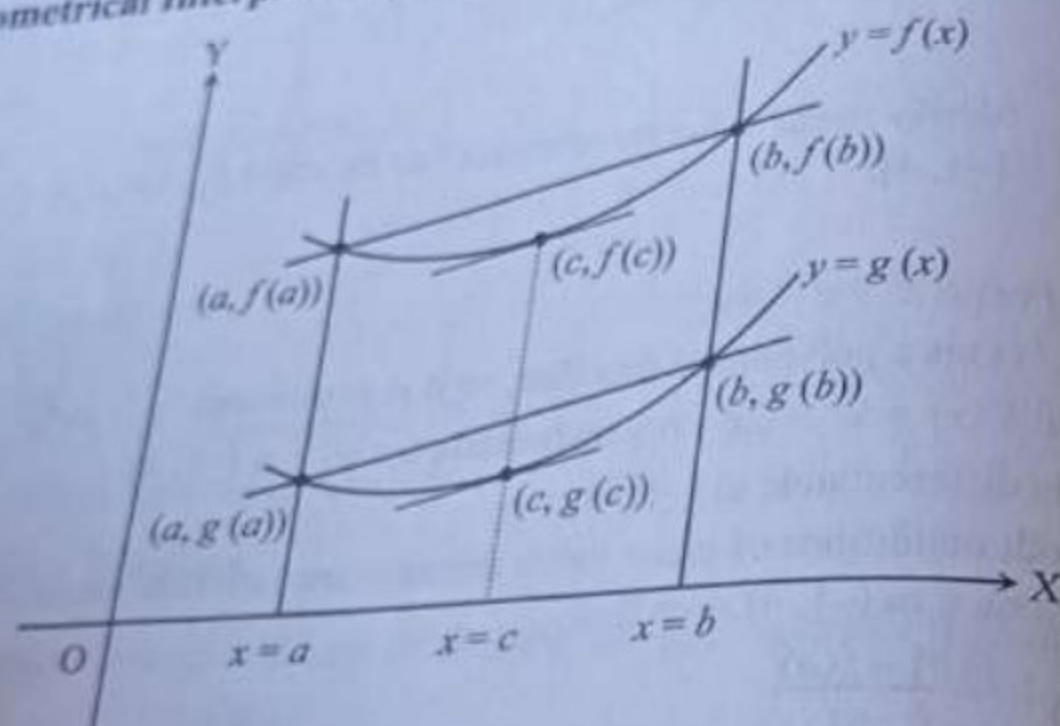
If f and g are two real valued functions such that

- f and g are continuous in the closed interval $[a, b]$
- f and g are differentiable in the open interval (a, b)
- $g'(x) \neq 0$ for all $x \in (a, b)$ then there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

This is also known as Cauchy's mean value theorem or Cauchy's formula or Second Mean Value Theorem of differential calculus. It is also called the Extended Mean value Theorem.

Geometrical Interpretation



By Cauchy's Mean Value Theorem,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\text{or, } \frac{\frac{f(b) - f(a)}{b - a}}{\frac{g(b) - g(a)}{b - a}} = \frac{f'(c)}{g'(c)}$$

$$\text{i.e. } \frac{\text{Slope of the chord joining } (a, f(a)) \text{ and } (b, f(b))}{\text{Slope of the chord joining } (a, g(a)) \text{ and } (b, g(b))} = \frac{\text{Slope of tangent at } (c, f(c))}{\text{Slope of tangent at } (c, g(c))}$$

Thus, the ratio of the slope of the chord joining $(a, f(a))$ and $(b, f(b))$ to the chord joining $(a, g(a))$ and $(b, g(b))$ is equal to the ratio of the slope of tangent at $(c, f(c))$ and tangent at $(c, g(c))$.

Example 4. Find the value of c by using Cauchy's Mean Value Theorem for the functions $f(x) = x^2$ and $g(x) = x^3$ in $[0, 1]$.

Solution

Here $f(x) = x^2$ and $g(x) = x^3$ are continuous on $[0, 1]$. Also, $f'(x) = 2x$ and $g'(x) = 3x^2$ exists for all x in $(0, 1)$. Moreover, $g'(x) = 3x^2 \neq 0$ for all x in $(0, 1)$. Hence all the conditions of Cauchy's Mean Value Theorem are satisfied. Hence, there exists a point c in $(0, 1)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{2c}{3c^2}$$

$$\text{or, } \frac{1 - 0}{1 - 0} = \frac{2}{3c}$$

$$\text{or, } 3c = 2$$

$$\therefore c = \frac{2}{3} \in (0, 1)$$

Thus, the required value of c is $\frac{2}{3}$.

Illustrative Examples

Example 1. Verify Rolle's theorem for the function $f(x) = \sqrt{1-x^2}$, $x \in [-1, 1]$.

Solution

Here, $f(x) = \sqrt{1-x^2}$, $x \in [-1, 1]$

For every value of x such that $-1 \leq x \leq 1$, $f(x)$ has a definite value, so $f(x)$ is continuous in $[-1, 1]$

Again, $f'(x) = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$ which exists for all x in $(-1, 1)$

So, $f(x)$ is differentiable in $(-1, 1)$

$$\text{Also, } f(1) = \sqrt{1-1^2} = 0$$

$$f(-1) = \sqrt{1-(-1)^2} = 0$$

$$f(1) = f(-1)$$

All the conditions of Rolle's theorem are satisfied. Hence, there exists at least one $c \in (-1, 1)$ such that $f'(c) = 0$

$$\Rightarrow \frac{-c}{\sqrt{1-c^2}} = 0$$

$$\Rightarrow c = 0 \in (-1, 1)$$

Hence Rolle's theorem is verified.

Example 2. Verify mean value theorem for the function $f(x) = (x-1)(x-2)(x-3)$ in $[1, 4]$.

Solution

Here,

$$f(x) = (x-1)(x-2)(x-3) \text{ in } [1, 4]$$

$$= x^3 - 6x^2 + 11x - 6$$

Since $f(x)$ is a polynomial function, so it is continuous in $[1, 4]$

Again, $f'(x) = 3x^2 - 12x + 11$ which exists for all $x \in (1, 4)$, so it is differentiable in $(1, 4)$.

Hence, both the conditions of mean value theorem are satisfied. So there exists at least one $c \in (1, 4)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\text{But } f(b) = f(4) = 4^3 - 6 \cdot 4^2 + 11 \cdot 4 - 6 = 6$$

$$f(a) = f(1) = 1^3 - 6 \cdot 1^2 + 11 \cdot 1 - 6 = 0$$

$$\therefore f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow 3c^2 - 12c + 11 = \frac{6 - 0}{4 - 1}$$

$$\Rightarrow 3c^2 - 12c + 11 = 2$$

$$\Rightarrow 3c^2 - 12c + 9 = 0$$

$$\Rightarrow c^2 - 4c + 3 = 0$$

$$\Rightarrow (c-1)(c-3) = 0$$

$$\Rightarrow c = 1, 3$$

But $c = 1 \notin (1, 4)$ and $c = 3 \in (1, 4)$

Hence mean value theorem is verified.

Example 3. Verify Rolle's theorem for the function $f(x) = \sin x$, $x \in [0, \pi]$. Also find a point in the curve represented by given function where the tangent is parallel to x-axis.

Solution

Here, $f(x) = \sin x$

For all $x \in [0, \pi]$, $f(x)$ has a definite value, so $f(x)$ is continuous in $[0, \pi]$

Again, $f(x) = \cos x$ which exists for all $x \in (0, \pi)$.

$\therefore f(x)$ is differentiable in $(0, \pi)$.

$$\text{Also, } f(0) = \sin 0 = 0$$

$$f(\pi) = \sin \pi = 0$$

$$\therefore f(0) = f(\pi)$$

$\therefore$ all the conditions of Rolle's theorem are satisfied.

Hence, there exists at least a point $c \in (0, \pi)$ such that $f'(c) = 0$

$$\text{or, } \cos c = 0 = \cos \frac{\pi}{2}$$

$$\therefore c = \frac{\pi}{2} \in (0, \pi)$$

Hence, Rolle's theorem is verified.

$c = \frac{\pi}{2}$ is the x-coordinate of the point at which tangent is parallel to x-axis.

Put $x = \frac{\pi}{2}$ in $y = f(x) = \sin x$, we have

$$y = \sin \frac{\pi}{2} = 1$$

$\therefore$ Required point is $\left(\frac{\pi}{2}, 1\right)$.

Example 4. Using Mean Value theorem, find a point on the parabola $y = (x-3)^2$, where the tangent is parallel to the chord joining the points $(3, 0)$ and $(4, 1)$.

Solution

The value of x ranges from 3 to 4.

So, $y = f(x) = (x-3)^2$, $x \in [3, 4]$

Since $f(x)$ is a polynomial function, it is continuous in $[3, 4]$.

Again, $f'(x) = 2x - 6$ which exists for all $x \in (3, 4)$.

So, $f(x)$ is differentiable in $(3, 4)$.

Both the conditions of mean value theorem are satisfied. Hence by mean value theorem, there exists at least one $c \in (3, 4)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\text{or, } f'(c) = \frac{f(4) - f(3)}{4 - 3}$$

$$\text{or, } 2c - 6 = \frac{(4-3)^2 - (3-3)^2}{1}$$

$$\text{or, } 2c - 6 = \frac{1}{1} = 1$$

$$\text{or, } 2c = 1 + 6$$

$$\therefore c = \frac{7}{2} \in (3, 4)$$

So, $c = \frac{7}{2}$ is the x-coordinate of the point at which the tangent is parallel to the chord joining the given points.

$$\text{Put } x = \frac{7}{2} \text{ in } y = (x-3)^2, \text{ we get } y = \left(\frac{7}{2} - 3\right)^2 = \frac{1}{4}$$

$\therefore$ Required point = $\left(\frac{7}{2}, \frac{1}{4}\right)$.

Exercise 3.4

- Verify Rolle's theorem for the following functions.
 - $f(x) = x^2$ in $[-1, 1]$
 - $f(x) = x^2 - 4$ in $-3 \leq x \leq 3$
 - $f(x) = x(x-2)^2$ in $[0, 2]$
 - $f(x) = \sin x$ in $[0, 2\pi]$
 - $f(x) = \sqrt{16-x^2}$ in $[-4, 4]$
- Using Rolle's theorem, find a point on the curve $f(x) = 3x - x^2$ in $[0, 3]$, where the tangent is parallel to x-axis.
- Verify Lagrange's mean value theorem for the following functions.
 - $f(x) = x^2 - 1$ in $[1, 3]$
 - $f(x) = x^2 + 2x - 1$ in $[0, 1]$
 - $f(x) = x^{\frac{2}{3}}$ in $[0, 1]$
 - $f(x) = \sqrt{x-1}$, $x \in [1, 3]$
 - $f(x) = e^x$, $x \in [0, 1]$
- Using mean value theorem, find the point on the curve $f(x) = (x-1)(x-2)(x-3)$, the tangent at the point is parallel to the chord joining the points $(1, 0)$ and $(4, 6)$.
- Find the value of c by using generalized mean value theorem (Cauchy's Mean Value Theorem) for the function $f(x) = x^2 + 3x - 2$ and $g(x) = x^2$ on $[1, 2]$.
- Verify the generalized mean value theorem for the functions
 - $f(x) = x^2 - 3x + 2$ and $g(x) = x^2 + 3x + 1$ on $[0, 1]$.
 - $f(x) = x^2$ and $g(x) = 2x + 3$ on $[1, 4]$.

Very Short Answer Questions

- State Rolle's theorem.
- If a function $f(x) = 1 - x^{\frac{2}{3}}$ is defined in the interval $[-1, 1]$ then find $f'(0)$.
- Find the value of c prescribed by Rolle's theorem for $f(x) = x^2 - 4$ in $[-3, 3]$.
- Write the geometrical interpretation of Lagrange's mean value theorem.
- State Cauchy's mean value theorem.

Answers

- $c = 0$
 - $c = 0$
 - $(\frac{3}{2}, \frac{9}{4})$
 - $c = 2$
 - $c = \log(e-1)$
- $\frac{3}{2}$
 - $\frac{1}{2}$
 - $\frac{5}{2}$
 - 0
- $c = 0$
 - $c = \frac{2}{3}$
 - $c = \frac{8}{27}$
 - $c = \frac{3}{2}$
- $\frac{1}{2}$
 - $\frac{5}{2}$
 - 0

Very Short Answer Questions

3.12 Taylor and Maclaurin Series

Definitions

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. The Taylor series generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n + \dots$$

The Maclaurin series generated by f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n + \dots$$

which is the Taylor series generated by f at $x = 0$.

Example 1. Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$. Find, where, if any, does the series converge to $\frac{1}{x}$?

Solution

$$\text{Here, } f(x) = \frac{1}{x} = x^{-1}, \quad f(2) = \frac{1}{2}$$

$$f'(x) = -x^{-2} \quad f'(2) = -2^{-2} = -\frac{1}{2^2}$$

$$f''(x) = 2x^{-3} = 2! x^{-3}, \quad \frac{f''(2)}{2!} = 2^{-3} = \frac{1}{2^3}$$

$$f^{(n)}(x) = (-1)^n n! x^{-(n+1)}$$

$$\frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}$$

The Taylor series of $f(x) = \frac{1}{x}$ at $x = 2$ is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \dots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \dots$$

$$= \frac{1}{2} - \frac{x-2}{2^2} + \frac{(x-2)^2}{2^3} - \dots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \dots$$

This is a geometric series with first term $\frac{1}{2}$ and ratio $(r) = \frac{-(x-2)}{2} = -\frac{(x-2)}{2}$

$$= \frac{-(x-2)}{2}$$

It converges absolutely for $|r| < 1$ i.e. $|x-2| < 2$
 i.e. $-2 < x-2 < 2$
 i.e. $0 < x < 4$

And its sum is $\frac{a}{1-r} = \frac{\frac{1}{2}}{1 + \frac{(x-2)}{2}}$

$$= \frac{1}{2 + (x-2)} = \frac{1}{x}$$

Hence the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$ converges to $\frac{1}{x}$ for $0 < x < 4$.

Taylor Polynomials

Let f be a function with derivatives of order k for $k = 1, 2, \dots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N , the Taylor Polynomial of order n generated by f at $x = a$ is the polynomial.

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example 2. Find the Taylor series and Taylor polynomials generated by $f(x) = e^x$ at $x = 0$.

Solution

$$f(x) = e^x, \quad f(0) = e^0 = 1$$

$$f'(x) = e^x, \quad f'(0) = e^0 = 1$$

$$f''(x) = e^x, \quad f''(0) = e^0 = 1$$

$$f^{(n)}(x) = e^x, \quad f^{(n)}(0) = 1$$

The Taylor series generated by f at $x = 0$ is

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

This is the exponential series. By definition it is also the Maclaurin series for e^x . The Taylor polynomial of order n at $x = 0$ is

$$P_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

3.13 Taylor's Theorem

If f and its first n derivatives $f', f'', \dots, f^{(n)}$ are continuous on $[a, b]$ or $[b, a]$ and $f^{(n)}$ is differentiable on (a, b) or (b, a) , then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

Corollary to Taylor's Theorem

If f has derivatives of all orders in an open interval I containing a , then for each positive integer n and for each x in I ,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

$$\text{where, } R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \text{ for some } c \text{ between } a \text{ and } x. \dots \text{(ii)}$$

Equation (i) is called Taylor's formula. The function $R_n(x)$ is called the remainder of order n . If $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for all x in I , then we say that the Taylor series generated by f at $x = a$ converges to f on I , and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Illustrative Examples

Example 1. Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x .

Solution

The function $f(x) = e^x$ has derivatives of all orders throughout the interval $I = (-\infty, \infty)$.

Then $f(x) = e^x$ and $a = 0$ give

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + R_n(x) \quad [\text{From previous example of } e^x]$$

and $R_n(x) = \frac{e^c}{(n+1)!} x^{n+1}$ for some c between 0 and x .

Since e^x is an increasing function of x , e^c lies between $e^0 = 1$ and e^x . When x is negative, so is c and $e^c < 1$.

When $x = 0$, $e^c = 1$ and $R_n(x) = 0$. When x is positive, so is c , and $e^c < e^x$. Thus,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \quad \text{when } x \leq 0,$$

and

$$|R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \quad \text{when } x > 0.$$

$$\text{Since, } \lim_{n \rightarrow \infty} \frac{x^{n+1}}{(n+1)!} = 0 \text{ for every } x,$$

So, $\lim_{n \rightarrow \infty} R_n(x) = 0$ and the series converges to e^x for every x .

$$\text{Thus, } e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^k}{k!} + \dots$$

Example 2. Find the Maclaurin series for the function $f(x) = \sin x$.

Solution

$f(x) = \sin x$	$f(0) = \sin 0 = 0$
$f'(x) = \cos x$	$f'(0) = \cos 0 = 1$
$f''(x) = -\sin x$	$f''(0) = -\sin 0 = 0$
$f'''(x) = -\cos x$	$f'''(0) = -\cos 0 = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = \sin 0 = 0$
...	...

The derivatives of $f(x)$ repeat in a cycle of 4 (i.e. 0, 1, 0, -1). This means that every derivative of even degree gives 0 and that of odd degree alternate between 1 and -1.

Thus, the required Maclaurin series of $f(x) = \sin x$ is

$$\begin{aligned} f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots \\ = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4 + \dots \\ = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

Exercise 3.5

1. Find the Taylor series for the following function at $x = 0$. (Maclaurin series)

a. e^{-x}

b. $\frac{1}{1+x}$

c. $\cos x$

2. Find the Taylor series expansion for $f(x) = e^x$ at

a. $x = 1$

b. $x = 2$

Very Short Answer Questions

- Write the expression for Maclaurin series.
- When does Taylor series become Maclaurin series?
- Write the Taylor series for $f(x) = e^x$ at $x = 0$.
- Write the Taylor series for $f(x) = e^{-x}$ at $x = 0$.
- Write the remainder of order n in Taylor's formula.

Answers

1. a. $\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$

b. $\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \dots$

c. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

2. a. $\sum_{n=0}^{\infty} \frac{e}{n!} (x-1)^n$

b. $\sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n$

3.14 Application of Derivatives in Economics

In business and economics, the vocabulary for rates of change and derivatives are **marginals**. To perform marginal analysis on cost, revenue or profit, we find the derivative of cost, revenue or profit. The derivatives of these quantities are called marginal cost, revenue, profit respectively.

Average Cost and Marginal Cost

If $C(x)$ be the total cost of producing and marketing x units of a product, then average cost (AC) is defined by

$$AC = \frac{C}{x};$$

and the marginal cost (MC) is defined to be the rate of change of C with respect to x . Thus

$$MC = \frac{dC}{dx}$$

For example, let $C(x)$ denote the cost needed to produce x units of goods in a day. It costs more to produce $(x + h)$ units, and the cost difference divided by h , is the average increase in cost per unit per day.

$$\text{Then, Marginal cost of production} = \frac{dC}{dx} = \lim_{h \rightarrow 0} \frac{C(x+h) - C(x)}{h}$$

Sometimes the marginal cost of production is defined to be the extra cost of producing one additional unit of output.

$$\frac{\Delta C}{\Delta x} = \frac{C(x+1) - C(x)}{1}$$

This is approximately the value of $\frac{dC}{dx}$.

Average Revenue and Marginal Revenue

If x be the quantity sold and P be the price (demand function), then the revenue function $R(x) = P \cdot x$.

The average revenue (AR) is defined by $AR = \frac{R}{x}$.

Since, $R = P \cdot x$, it follows that $AR = \frac{R}{x} = \frac{P \cdot x}{x} = P$.

Thus, average revenue is same as the price per unit.

The marginal revenue is defined as the rate of change of total revenue function with respect to quantity demanded. Hence, marginal revenue is the derivative of R with respect to x .

$$\text{Marginal revenue (MR)} = \frac{dR}{dx}$$

It is interpreted as the approximate revenue received from selling one additional unit of output.

Profit Function and Marginal Profit

The profit function (π) is defined by $\pi(x) = R(x) - C(x)$. The marginal profit is the derivative of profit function with respect to x .

3.15 Price and Output

Price: The amount of money which has to be paid to acquire any product.

Output: The quantity of goods or services produced in a given time period by a company whether consumed or used for further production is called output. Generally, we denote the output by x or Q .

3.16 Competitive Equilibrium of Firm

In competitive equilibrium, price is equal to the short run marginal cost. So, no firm can sell an extra unit at a price that cover all its short run marginal cost.

A firm is said to be in equilibrium when it has no incentive either to expand or to contract its output.

The important conditions of equilibrium of firm are:

- Making a maximum profit
- Incurring a minimum loss

At equilibrium price, the quantity supplied is equal to quantity demanded. A firm is in equilibrium when it maximizes the profit. When a firm is in equilibrium, there is no reason to increase or decrease the output.

There are two types of equilibrium: short run and long run. There are fixed and variable factors in short run and there are no fixed factors in long run.

Condition for Cost Minimization and Minimum Average Cost

Let $C = C(Q)$ be the given cost function. Then,

$$\text{Average cost (AC)} = \frac{C}{Q}.$$

The cost (C) is minimum when

$$\frac{d(AC)}{dQ} = 0 \text{ and } \frac{d^2(AC)}{dQ^2} > 0$$

Condition for Revenue Maximization

If $R = R(Q)$ be given revenue function the R is maximum when

$$\frac{d(R)}{dQ} = 0 \text{ and } \frac{d^2(R)}{dQ^2} < 0.$$

Maximization of Total Profit

If $R(Q)$ and $C(Q)$ denote the total revenue function and total cost function respectively then profit function

$$\pi(Q) = R(Q) - C(Q)$$

We know that profit is maximum when

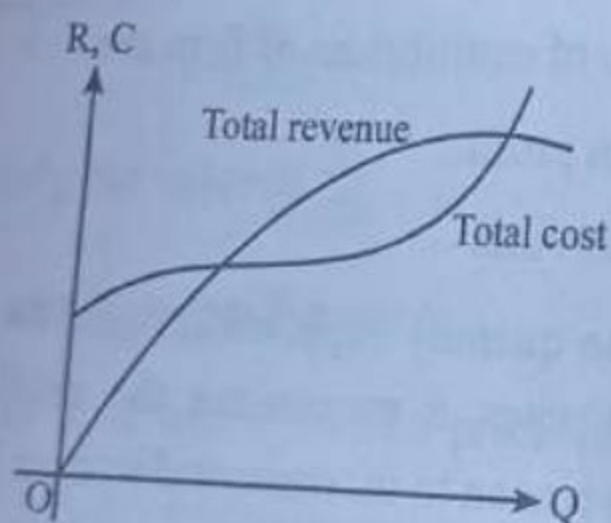
$$\frac{d\pi}{dQ} = 0 \text{ and } \frac{d^2\pi}{dQ^2} < 0.$$

$$\text{But, } \frac{d\pi}{dQ} = \frac{d}{dQ}(R - C) = \frac{dR}{dQ} - \frac{dC}{dQ}$$

$$\text{Set } \frac{d\pi}{dQ} = 0, \text{ we have } \frac{dR}{dQ} = \frac{dC}{dQ}.$$

i.e. Marginal revenue = Marginal cost

Thus, at the level of maximum profit, the slope of the tangent to the total cost curve must be equal to the tangent to the total revenue curve.



Thus, to maximize profit, it is necessary that $MR = MC$. For this, we must have,

$$\frac{d^2\pi}{dQ^2} < 0.$$

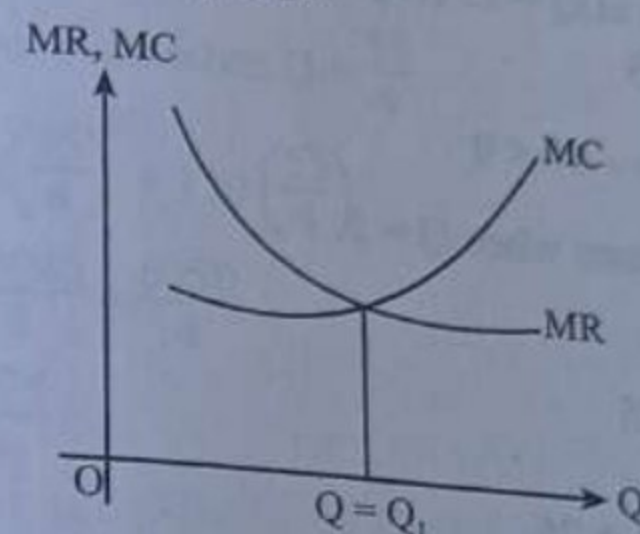
i.e. $\frac{d^2\pi}{dQ^2} < 0$

$$\text{or, } \frac{d}{dQ}(MR - MC) < 0$$

$$\text{or, } \frac{d}{dQ}(MR) - \frac{d}{dQ}(MC) < 0$$

$$\therefore \frac{d}{dQ}(MR) < \frac{d}{dQ}(MC).$$

This shows that the slope of the marginal revenue curve must be less than the slope of the marginal cost curve at the point of maximum profit. In other words, the marginal cost curve must cut the marginal revenue curve from below at the point of maximum profit.



Here the profit is maximum at $Q = Q_1$. Thus, $MR = MC$ is a rule to profit maximization for competitive equilibrium of firms.

Illustrative Examples

Example 1. The demand function for a certain commodity is $P = \frac{1}{3}Q^2 - 10Q + 75$. Find the value of Q and the corresponding value of P that maximizes the revenue.

Solution

We have

$$R = P \times Q$$

$$= \left(\frac{1}{3}Q^2 - 10Q + 75\right) \times Q$$

$$\therefore R = \frac{Q^3}{3} - 10Q^2 + 75Q$$

Then,

$$\frac{dR}{dQ} = \frac{3Q^2}{3} - 20Q + 75$$

$$= Q^2 - 20Q + 75$$

and,

$$\frac{d^2R}{dQ^2} = 2Q - 20$$

For maxima or minima, we have,

$$\frac{dR}{dQ} = 0$$

$$\text{or, } Q^2 - 20Q + 75 = 0$$

$$\text{or, } Q^2 - 15Q - 5Q + 75 = 0$$

$$\text{or, } Q(Q - 15) - 5(Q - 15) = 0$$

$$\therefore \text{ Either } Q = 15 \text{ or } 5$$

When $Q = 15$

$$\frac{d^2R}{dQ^2} = 2 \times 15 - 20 = 10 > 0.$$

So, R is minimum at $Q = 15$ (Rejecting)

Again, when $Q = 5$

$$\frac{d^2R}{dQ^2} = 2 \times 5 - 20 = -10 < 0$$

Hence, R is maximum when $Q = 5$.

When $Q = 5$,

$$P = \frac{1}{3}Q^2 - 10Q + 75$$

$$\text{or, } P = \frac{1}{3} \times 25 - 10 \times 5 + 75$$

$$\text{or, } P = \frac{25 - 150 + 225}{3}$$

$$\text{or, } P = \frac{100}{3}$$

$$\therefore P = \frac{100}{3}$$

Example 2. If the demand function $P = 104 - 3Q$ and cost function $C = -6Q + Q^2$, then find the maximum profit. Also find the output for minimum cost.

Solution

First Case

$$P = 104 - 3Q \text{ and } C = -6Q + Q^2$$

$$\therefore \text{ Revenue function (R)} = Q(104 - 3Q) = 104Q - 3Q^2$$

$$\text{Profit } (\pi) = R - C$$

$$= 104Q - 3Q^2 + 6Q - Q^2$$

$$= -4Q^2 + 110Q$$

$$\frac{d\pi}{dQ} = -8Q + 110$$

$$\frac{d^2\pi}{dQ^2} = -8$$

For maxima or minima, we have,

$$\frac{d\pi}{dQ} = 0$$

$$\text{or, } -8Q + 110 = 0$$

$$\therefore Q = \frac{55}{4}$$

$$\text{When } Q = \frac{55}{4},$$

$$\frac{d^2\pi}{dQ^2} = -8 < 0$$

Hence profit is maximum when $Q = \frac{55}{4}$

$$\begin{aligned} \text{Max. Profit} &= -4\left(\frac{55}{4}\right)^2 + 110\left(\frac{55}{4}\right) \\ &= \frac{-3025}{4} + \frac{6050}{4} \\ &= \frac{3025}{4} \end{aligned}$$

Second Case

$$C = -6Q + Q^2$$

$$\frac{dC}{dQ} = -6 + 2Q$$

$$\frac{d^2C}{dQ^2} = 2$$

For maxima or minima, we have,

$$\frac{dC}{dQ} = 0$$

$$\text{or, } -6 + 2Q = 0$$

$$\therefore Q = 3$$

$$\text{When } Q = 3, \frac{d^2C}{dQ^2} = 2 > 0.$$

Hence C is minimum.

For minimum cost, $Q = 3$.

Example 3. The cost function of a firm is given by the equation

$$C = 300x - 10x^2 + \frac{1}{3}x^4 \text{ where } C \text{ stands for cost and } x \text{ for output.}$$

Calculate the output in which the average cost is minimum.

Solution

$$\text{Here, } C = 300x - 10x^2 + \frac{1}{3}x^4$$

$$\text{Average cost (AC)} = \frac{300x - 10x^2 + \frac{1}{3}x^4}{x} = 300 - 10x + \frac{1}{3}x^3$$

$$\therefore \frac{d(AC)}{dx} = -10 + x^2$$

$$\frac{d^2(AC)}{dx^2} = 2x$$

For maxima and minima, we have,

$$\frac{d(AC)}{dx} = 0$$

$$-10 + x^2 = 0$$

$$\text{or, } x^2 = 10$$

$$\therefore x = \pm\sqrt{10}, x = -\sqrt{10} \text{ is not possible since output can't be negative}$$

$$\therefore x = \sqrt{10}$$

$$\text{At } x = \sqrt{10}$$

$$\frac{d^2(AC)}{dx^2} = 2\sqrt{10} > 0$$

Average cost is minimum when output $x = \sqrt{10}$.

Example 4. A firm produces x tons of output at the total cost $C = \text{Rs} \left(\frac{x^3}{10} - 5x^2 + 10x \right)$. At what level of output will the marginal cost and the average cost attain their respective minima?

Solution

Marginal Cost (MC) is given by

$$MC = \frac{dC}{dx} = \text{Rs.} \left(\frac{3x^2}{10} - 10x + 10 \right)$$

and average variable cost (AVC) is given by

$$AC = \left(\frac{\frac{1}{10}x^3 - 5x^2 + 10x}{x} \right) = \frac{1}{10}x^2 - 5x + 10$$

Now,

$$\text{Let } y = \frac{3x^2}{10} - 10x + 10$$

$$\frac{dy}{dx} = \frac{3x}{5} - 10$$

$$\frac{d^2y}{dx^2} = \frac{3}{5}$$

For maxima or minima, we have,

$$\frac{dy}{dx} = 0$$

$$\text{or, } \frac{3x}{5} - 10 = 0$$

$$\text{or, } x = \frac{50}{3}$$

At $x = \frac{50}{3}$ then

$$\frac{d^2y}{dx^2} = \frac{3}{5} > 0$$

Hence MC attains its minima at $x = \frac{50}{3}$

Similarly,

$$\frac{d(AC)}{dx} = \frac{1}{5}x - 5$$

$$\frac{d^2(AC)}{dx^2} = \frac{1}{5}$$

For maxima or minima, we have,

$$\frac{d(AC)}{dx} = 0$$

$$\frac{1}{5}x - 5 = 0$$

$$x = 25$$

At $x = 25$,

$$\frac{d^2(AC)}{dx^2} = \frac{1}{5} > 0$$

Here AC attains its minima at $x = 25$.

Example 5. If demand function and cost function are given by $P(Q) = 1 - 3Q$ and $C(Q) = Q^2 - 2Q$ respectively, where Q is the quantity (number) of the product then find output of the factory for the maximum profit.

Solution

Cost function, $C(Q) = Q^2 - 2Q$

Demand function, $P(Q) = 1 - 3Q$

$$\begin{aligned} \therefore \text{Revenue function, } R(Q) &= \text{Demand function} \times \text{Quantity} \\ &= (1 - 3Q) \times Q \\ &= Q - 3Q^2 \end{aligned}$$

We have,

$$\begin{aligned} \text{Profit function, } \pi(Q) &= R(Q) - C(Q) \\ &= Q - 3Q^2 - (Q^2 - 2Q) \\ &= Q - 3Q^2 - Q^2 + 2Q \\ &= 3Q - 4Q^2 \end{aligned}$$

Differentiating both sides w.r. to Q , we get,

$$\frac{d}{dQ}(\pi(Q)) = \frac{d}{dQ}(3Q - 4Q^2)$$

$$\text{or, } \pi'(Q) = 3 - 8Q$$

Again, differentiating both sides w.r. to Q , we get,

$$\pi''(Q) = -8.$$

For maximum and minimum values, we have,

$$\pi'(Q) = 0$$

$$\text{or, } 3 - 8Q = 0$$

$$\text{or, } 3 = 8Q$$

$$\therefore Q = \frac{3}{8}.$$

When $Q = \frac{3}{8}$, $\pi''(Q) = -8$ which is negative. Hence, profit $\pi(Q)$ is maximum

$$\text{at } Q = \frac{3}{8}.$$

Exercise 3.6

- If the total cost is given by $C = 3Q^2 + 5Q + 50$, find the average cost and marginal cost and their values at the production level 6.
 - The demand function for a commodity is given by $P = 13 - 3x$. Find the average revenue and marginal revenue.
 - If the demand function, $P = 30 - 4x$ and total cost function, $C = 35x + 600$. Find the marginal profit.
- A firm estimates that its daily total cost function is $C(x) = x^3 - 6x^2 + 13x + 15$ and its total revenue function is $R(x) = 28x$. Find the value of x that maximizes the daily profit.
- If the revenue function is $R = Q - 3Q^2$ and the cost function $C = Q^2 - 2Q$, find the value of the maximum profit.
- A firm has a demand function $P = 108 - 5Q$ and cost function $C = -12Q + Q^2$. Find the price at which the profit is maximum. Find the maximum profit.
- The demand equation for a certain commodity is $P = \frac{1}{3}Q^2 - \frac{15}{2}Q + 50$. Find the value of Q and the corresponding value of P that maximizes the revenue.
- The demand function for a certain commodity is $P = \frac{Q^2}{3} - 10Q + 75$. Find the value of Q and the corresponding value of P that maximizes the revenue.
- Given the demand function $P = 20 - Q$ and the total cost function $C = Q^2 + 8Q + 2$, determine the optimal output Q , price P and total profit under profit maximization.

- Given the demand function $P = 20 - Q$ and total cost function $C = Q^2 + 8Q$, determine the price at which the profit is maximum. Also find the maximum profit.

- A company produces Q units of output at a total cost of $C = \frac{1}{4}Q^2 + 3Q + 100$. Verify that the minimum average cost is equal to the marginal cost at the level minimizing the average cost.

- A company produces Q units of output at a total cost of $\frac{1}{3}Q^3 - 18Q^2 + 160Q$. Find

- the output at which marginal cost is minimum,
- the output at which average cost is minimum, and
- the output at which average cost is equal to marginal cost.

Very Short Answer Questions

- If the total cost function is $C = x^2 + 3$, then find the average cost function.
- If the total cost function of a company is $C = x^2 + 30x + 500$ then find marginal cost.
- If x is the units of sales of a product, P is the price per unit and R is the revenue then find marginal revenue (MR).
- If the demand function of a product is $P = 200 - 5Q$ then find marginal revenue.
- If the revenue function is $R = Q - 5Q^2$ and cost function is $C = Q^2 - 4Q$ then find marginal profit function.

Answers

- $AC = 3Q + 5 + \frac{50}{Q}$, 31.33, $MC = 6Q + 5$, 41
 - $AR = 13 - 3x$, $MR = 13 - 6x$
 - $MP = -8x - 5$
- $x = 5$
- $\frac{9}{16}$
- 58 and 600
- $5, \frac{125}{6}$
- $P = \frac{100}{3}$ at $Q = 5$
- $Q = 3$, $P = 17$, $\pi = 16$.
- 17 and 18
- 18
 - 27
 - 27

Very Short Answer Questions

- $x + \frac{3}{x}$
- $2x + 30$
- $P + x \frac{dP}{dx}$
- $200 - 10Q$
- $-12Q + 5$

3.17 Rate Measures

3.17.1 The Derivative as Rate of Change

Let $y = f(x)$ be a continuous function. If Δx and Δy be the small increments in x and y respectively, then

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

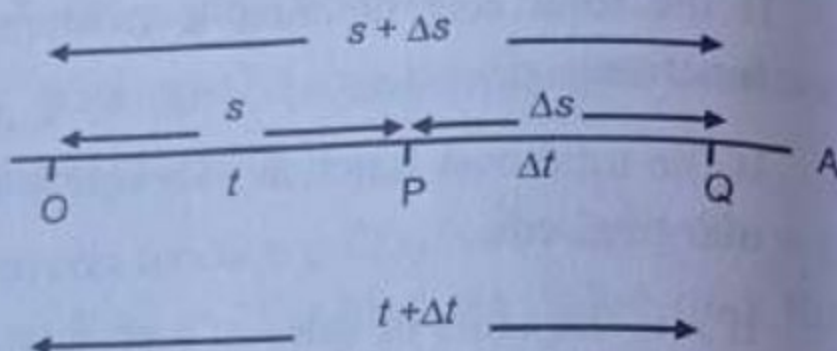
represents the slope of the secant line to the curve $y = f(x)$. This can be regarded as the average rate of change of y with respect to x in the interval from x to $x + \Delta x$. At the point $x = a$,

$$f'(a) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \frac{dy}{dx}$$

is called the instantaneous rate of change of y with respect to x at $x = a$, if this limit exists.

3.17.2 Physical Interpretation of Derivative

Let us suppose that a particle be moving along a straight line OA and pass the points P and Q at the times t and $t + \Delta t$ respectively. Let s and $s + \Delta s$ be the distances described in times t and $t + \Delta t$ respectively, from the fixed point O. Now $s = f(t)$ is the equation of the law of motion.



If the ratio $\frac{\Delta s}{\Delta t}$ is constant, i.e., the equal distances are described in equal intervals of time, then the motion is called uniform, and the ratio is called the velocity at any instant. If the motion is not uniform, the ratio $\frac{\Delta s}{\Delta t}$ varies as Δt varies, and the ratio can be taken as the velocity at an instant. In such case, the ratio is defined as the average velocity of the particle during the interval Δt . Thus,

$$\text{Average velocity} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t},$$

and $v = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$ is the instantaneous velocity of the particle at the time t . Similarly, if Δv is the change in velocity of the particle as it moves from P

to Q during the time interval Δt , then $a = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ is the instantaneous acceleration of the particle at time t .

Illustrative Examples

Example 1. Find the velocity of a rectilinear motion given by the equation $s = 3t^2 - 2t + 1$ at the instant when $t = 2$ seconds.

Solution

We have $s = 3t^2 - 2t + 1$. Then

$$\frac{ds}{dt} = 6t - 2$$

$$\text{At } t = 2,$$

$$\frac{ds}{dt} = 6 \times 2 - 2 = 10 \text{ m/sec.}$$

Example 2. The side of a square sheet is increasing at the rate of 5 cm/min. At what rate is the area increasing when the side is 12 cm long?

Solution

Let x be the length of side and A be the area of the square at time t . By question,

$$\frac{dx}{dt} = 5 \text{ cm/min}$$

$$\text{When } x = 12 \text{ cm, } \frac{dA}{dt} = ?$$

We have,

$$A = x^2$$

$$\frac{dA}{dt} = 2x \frac{dx}{dt}$$

$$\text{When } x = 12 \text{ cm,}$$

$$\frac{dA}{dt} = 2 \times 12 \times 5$$

$$\therefore \frac{dA}{dt} = 120 \text{ cm}^2/\text{min}$$

Example 3. A stone thrown into a pond produces circular ripples which expand from the point of impact. If the radius of the ripple increases at the rate of 3.5 cm/sec, how fast is the area growing when the radius is 15 cm? ($\pi = \frac{22}{7}$)

Solution

Let r be the radius and A be the area of a circular ripple in time t . By given,

$$\frac{dr}{dt} = 3.5 \text{ cm/sec}$$

3.17 Rate Measures

3.17.1 The Derivative as Rate of Change

Let $y = f(x)$ be a continuous function. If Δx and Δy be the small increments in x and y respectively, then

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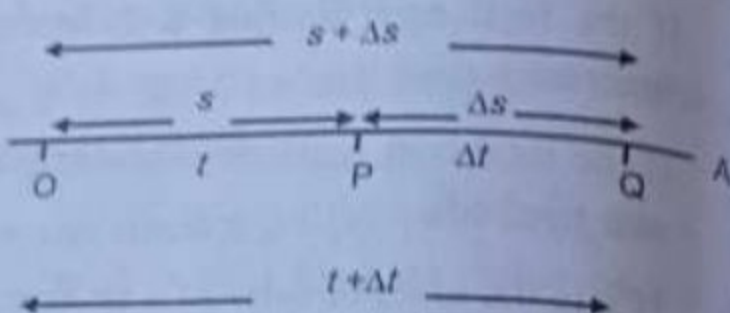
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Illustrative Examples

Example 1. Find the velocity of a rectilinear motion given by the equation $s = 3t^2 - 2t + 1$ at the instant when $t = 2$ seconds.

Solution We have $s = 3t^2 - 2t + 1$. Then

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$$\text{At } t = 2,$$

$$\frac{ds}{dt} = 6 \times 2 - 2$$

$$= 10 \text{ m/sec.}$$

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Solution

Let x be the length of side and A be the area of the square at time t . By question,

$$\frac{dx}{dt} = 5 \text{ cm/min}$$

$$\text{When } x = 12 \text{ cm, } \frac{dA}{dt} = ?$$

We have,

$$A = x^2$$

$$\frac{dA}{dt} = 2x \frac{dx}{dt}$$

$$\text{When } x = 12 \text{ cm,}$$

$$\frac{dA}{dt} = 2 \times 12 \times 5$$

$$\therefore \frac{dA}{dt} = 120 \text{ cm}^2/\text{min}$$

Example 3. A stone thrown into a pond produces circular ripples which expand from the point of impact. If the radius of the ripple increases at the rate of 3.5 cm/sec, how fast is the area growing when the radius is 15 cm? $\left(\pi = \frac{22}{7}\right)$

Solution

Let r be the radius and A be the area of a circular ripple in time t . By given,

$$\frac{dr}{dt} = 3.5 \text{ cm/sec}$$

We have,

$$A = \pi r^2$$

$$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

When $r = 15$ cm, then

$$\begin{aligned} \therefore \frac{dA}{dt} &= \pi \times 2 \times 15 \times 3.5 \\ &= \frac{22}{7} \times 30 \times 3.5 \\ &= 330 \text{ cm}^2/\text{sec} \end{aligned}$$

Example 4. A spherical balloon is inflated at the rate of 10 cubic cm/sec. At what rate is the radius increasing when the radius is 10 cm.

Solution

Let r be the radius and V be the volume of the spherical balloon at time t .

Then,

$$\frac{dV}{dt} = 10 \text{ cm}^3/\text{sec}$$

When $r = 10$ cm, $\frac{dr}{dt} = ?$

We have,

$$V = \frac{4}{3} \pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3} \cdot \pi \cdot 3r^2 \cdot \frac{dr}{dt}$$

$$\text{or, } \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

When $r = 10$ cm,

$$10 = 4\pi \times 10^2 \times \frac{dr}{dt}$$

$$\text{or, } \frac{1}{40\pi} = \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{1}{40\pi} \text{ cm/sec.}$$

Example 5. Water flows into an inverted conical tank at the rate of $27 \text{ ft}^3/\text{min}$. When the depth of water is 2 ft, how fast is the level rising? Assume that the height of the tank is 4 ft and the radius of the top is 1 ft.

Solution

Let ABC be the conical tank into which water is flowing at the rate of $27 \text{ ft}^3/\text{min}$. Let h be the height AE of the water surface and r be the radius EF of the water surface at time t . Now $\triangle ACD$ and $\triangle AFE$ are similar. So,

$$\frac{AE}{AD} = \frac{EF}{DC}$$

$$\text{or, } \frac{h}{4} = \frac{r}{1}$$

$$\therefore r = \frac{h}{4}$$

Let V be the volume of water in the tank. We have,

$$\begin{aligned} V &= \frac{1}{3} \pi r^2 h \\ &= \frac{1}{3} \pi \left(\frac{h}{4}\right)^2 h \\ &= \frac{1}{48} \pi h^3 \end{aligned}$$

$$\begin{aligned} \frac{dV}{dt} &= \frac{1}{48} \pi \cdot 3h^2 \cdot \frac{dh}{dt} \\ &= \frac{\pi}{16} \cdot h^2 \cdot \frac{dh}{dt} \end{aligned}$$

When $h = 2$ ft,

$$27 = \frac{\pi}{16} \times 2^2 \times \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{108}{\pi} \text{ ft/min.}$$

Example 6. Two concentric circles are expanding in such a way that the radius of the inner circle is increasing at the rate of 6 cm/sec and that of the outer circle at the rate of 2.5 cm/sec. At a certain time the radius of the inner and the outer circles are respectively 20 cm and 32 cm. At that time how fast is the area between the circle increasing or decreasing?

Solution

Let r_1 be the radius of inner circle and r_2 be the radius of the outer circles at time t . By question,

$$\frac{dr_1}{dt} = 6 \text{ cm/sec and}$$

$$\frac{dr_2}{dt} = 2.5 \text{ cm/sec}$$

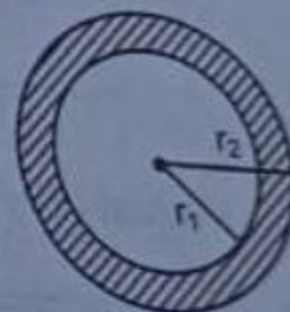
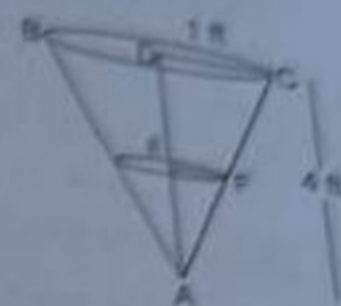
Let A be the area between the circles at time t . Then,

$$\begin{aligned} A &= \text{Area of outer circle} - \text{Area of inner circle} \\ &= \pi r_2^2 - \pi r_1^2 \end{aligned}$$

$$\frac{dA}{dt} = \pi \cdot 2r_2 \cdot \frac{dr_2}{dt} - \pi \cdot 2r_1 \cdot \frac{dr_1}{dt}$$

$$\text{or, } \frac{dA}{dt} = 2\pi r_2 \times 2.5 - 2\pi r_1 \times 6$$

$$= 5\pi r_2 - 12\pi r_1$$



When $r_1 = 20$ cm and $r_2 = 32$ cm

$$\begin{aligned}\frac{dA}{dt} &= 5\pi \times 32 - 12\pi \times 20 \\ &= 160\pi - 240\pi \\ &= -80\pi\end{aligned}$$

Hence, the area between the circles is decreasing at the rate of 80π cm²/sec.

Example 7. A point is moving along the curve $y = 2x^3 - 3x^2$ in such a way that its x -coordinate is increasing at the rate of 4 ft/sec. Find the rate at which the distance of the point from the origin is increasing when the point is at (2, 4).

Solution

Let $P(x, y)$ be a point moving along the curve $y = 2x^3 - 3x^2$. By given,

$$\frac{dx}{dt} = 4 \text{ ft/sec.}$$

Again,

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}(2x^3 - 3x^2) \\ &= (6x^2 - 6x) \frac{dx}{dt} \\ &= (6x^2 - 6x) \times 4 \\ &= 24x(x - 1)\end{aligned}$$

When $(x, y) = (2, 4)$,

$$\begin{aligned}\frac{dy}{dt} &= 24 \times 2(2 - 1) \\ &= 48\end{aligned}$$

Let S be the distance between the origin $O(0, 0)$ and the point $P(x, y)$. Then,

$$\begin{aligned}S &= \sqrt{x^2 + y^2} \\ \frac{dS}{dt} &= \frac{d}{dt}(x^2 + y^2)^{1/2} \\ &= \frac{1}{2\sqrt{x^2 + y^2}} \left(2x \frac{dx}{dt} + 2y \frac{dy}{dt} \right) \\ &= \frac{1}{\sqrt{x^2 + y^2}} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right)\end{aligned}$$

When $(x, y) = (2, 4)$

$$\begin{aligned}\frac{dS}{dt} &= \frac{1}{\sqrt{2^2 + 4^2}} (2 \times 4 + 4 \times 48) \\ &= \frac{1}{\sqrt{20}} (200) \\ &= \frac{1}{2\sqrt{5}} \times 200\end{aligned}$$

$$\therefore \frac{dS}{dt} = 20\sqrt{5} \text{ ft/sec.}$$

Exercise 3.7

- The distance s , in meters, travelled in t seconds by a particle moving in a straight line is given by $s = t^3 - 2t^2$. Find the velocity and acceleration of the particle when $t = 2$ seconds.
- Find the rate of change of the volume of a cylinder of radius r and height h will respect to a change in the radius.
- The radius of a circular plate is increasing at the rate of 0.20 cm/sec. At what rate is the area increasing when the radius of the plate is 25 cm?
- A spherical balloon is inflated at the rate of 10 cubic cm/sec. At what rate is the radius increasing when the radius is 6 cm.
 - The radius of a sphere is increasing at a rate of 1 cm/sec. Find the rate of change of volume at this time when the radius is 3 cm.
- A spherical ball of salt is dissolving in water in such a way that the rate of decrease in volume at any instant is proportional to the surface. Prove that the radius is decreasing at the constant rate.
- The side of a square is increasing at the rate of 0.2 cm/sec. Find the rate of increase of the (a) perimeter of the square and (b) area of square when the side of the square is 12 cm.
- If the volume of an expanding cube is increasing at the rate of 4 ft³/min, how fast is its surface area increasing when the surface area is 24 sq.ft.?
- Water flows into an inverted conical vessel at the rate of 24 m³/m. When the depth of water is 4 m, how fast is the level rising, assuming that the height of the vessel is 8 m and the radius at the top is 2 m?
- Two concentric circles are expanding in such a way that the radius of the inner circle is increasing at the rate of 10 cm/sec and that of the outer circle at the rate of 7 cm/sec. At a certain time, the radii of the inner and the outer circles are respectively 24 cm and 30 cm. At that time, is the area between the circles increasing or decreasing?
- A 1.6 m man walks away from a 10 m lamp post at the rate of 3 m/sec. How fast is his shadow lengthening when he is 18 m from the post?
- Sand is pouring from a pipe at the rate of 12 cm³/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand-cone increasing when the height is 4 cm?

12. A kite is 24 m high and there are 25 m of chord out. If the kite moves horizontally at the rate of 36 km/h. directly away from the person who is flying it, how fast is the chord paid out?

Very Short Answer Questions

- Find the velocity of a rectilinear motion $f(t)$ in meters given by $f(t) = 4t^2 - 3t + 1$ at $t = 4$ sec.
- The radius of expanding circular plate is 1.2 cm/sec. Find the rate of increase of its circumference.

Answers

- 4 m/sec, 8 m/sec²
- $2\pi rh$
- $10\pi \text{ cm}^2/\text{sec}$
- a. $\frac{5}{72\pi} \text{ cm/sec}$ b. $36\pi \text{ cu cm/sec}$
- a. 0.8 cm/sec b. $4.8 \text{ cm}^2/\text{sec}$
- $8 \text{ ft}^2/\text{min}$
- $\frac{24}{\pi} \text{ m/min}$
- $60\pi \text{ cm}^2/\text{sec}$ (Decreasing)
- $\frac{4}{7} \text{ m/sec}$
- $\frac{1}{48\pi} \text{ cm/sec}$
- 10.08 km/h.

Very Short Answer Questions

- 29 m/s
- $2.4\pi \text{ cm/sec}$

Anti-Derivative and Its Applications

UNIT

4

4.1 Introduction

The reverse process of differentiation is antiderivative or integration. To find the area between the curves and coordinate axes, the integral calculus was developed. Here, we have to find a function whose derivative is known. This inverse process is integration.

$$f \xrightarrow{\text{differentiation}} f' \text{ (derivative)}$$

$$\text{(integral)} f \xleftarrow{\text{integration}} f'$$

Indefinite Integral

Consider the expression $\frac{d}{dx}(3x^2 + 5) = 6x$.

Thus the derivative of $3x^2 + 5$ with respect to x is $6x$ or $3x^2 + 5$ is a function whose derivative is $6x$. Therefore, we can say that integration of $6x$ is $3x^2 + 5$.

Again, consider the followings:

$$\frac{d}{dx}(3x^2 - 10) = 6x$$

$$\frac{d}{dx}\left(3x^2 + \frac{7}{2}\right) = 6x$$

$$\frac{d}{dx}\left(3x^2 - \frac{1}{2}\right) = 6x$$

Thus, we can see that the function $f(x) = 6x$ has several antiderivatives. It can be shown that any antiderivative of $6x$ must have the form $3x^2 + c$. This leads to following definition of indefinite integral.

Definition

A function $F(x)$ is antiderivative of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f . The set of all antiderivatives of f is the **indefinite integral** of f with respect to x , denoted by

$$\int f(x) dx$$

The symbol $\int$ is an **integral sign**.

The function f is the **integrand** of the integral and x is the **variable of integration**.

From above example and definition, we can write.

$$\int f(x) dx = F(x) + c$$

Integration sign
Integrand
an antiderivative of $f(x)$
constant of integration (Arbitrary constant)

So, we can write the above example in notation form as

$$\int 3x^2 dx = 3x^3 + c.$$

Many of indefinite integrals needed in scientific work are found by reversing the derivative formulae. Some important integral formulas are given below.

Indefinite Integral	Reversed derivative formula
1. $\int 1 dx = x + c$	$\frac{d}{dx}(x + c) = 1$
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$
3. $\int \frac{1}{x} dx = \ln x + c$	$\frac{d}{dx}(\ln x + c) = \frac{1}{x}$
4. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a} + c, n \neq -1$	$\frac{d}{dx}\left(\frac{(ax+b)^{n+1}}{(n+1)a} + c\right) = (ax+b)^n$
5. $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln(ax+b) + c$	$\frac{d}{dx}\left(\frac{1}{a} \ln(ax+b) + c\right) = \frac{1}{ax+b}$
6. $\int e^x dx = e^x + c$	$\frac{d}{dx}(e^x + c) = e^x$
7. $\int e^{ax} dx = \frac{e^{ax}}{a} + c$	$\frac{d}{dx}\left(\frac{e^{ax}}{a} + c\right) = e^{ax}$
8. $\int a^x dx = \frac{a^x}{\ln a} + c$	$\frac{d}{dx}\left(\frac{a^x}{\ln a} + c\right) = a^x$

Integrals of Trigonometric Functions

Indefinite integral	Reversed derivative formula
1. $\int \sin x dx = -\cos x + c$	$\frac{d}{dx}(-\cos x + c) = \sin x$
2. $\int \cos x dx = \sin x + c$	$\frac{d}{dx}(\sin x + c) = \cos x$
3. $\int \sec^2 x dx = \tan x + c$	$\frac{d}{dx}(\tan x + c) = \sec^2 x$
4. $\int \operatorname{cosec}^2 x dx = -\cot x + c$	$\frac{d}{dx}(-\cot x + c) = \operatorname{cosec}^2 x$
5. $\int \sec x \tan x dx = \sec x + c$	$\frac{d}{dx}(\sec x + c) = \sec x \tan x$
6. $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$	$\frac{d}{dx}(-\operatorname{cosec} x + c) = \operatorname{cosec} x \cot x$
7. $\int \sin ax dx = -\frac{\cos ax}{a} + c$	$\frac{d}{dx}\left(-\frac{\cos ax}{a} + c\right) = \sin ax$
8. $\int \cos ax dx = \frac{\sin ax}{a} + c$	$\frac{d}{dx}\left(\frac{\sin ax}{a} + c\right) = \cos ax$

Examples

$$1. \int x^4 dx = \frac{x^{4+1}}{4+1} + c = \frac{x^5}{5} + c$$

$$2. \int \frac{1}{\sqrt{x}} dx = \int \frac{1}{x^{\frac{1}{2}}} dx = \int x^{-\frac{1}{2}} dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 2x^{\frac{1}{2}} + c = 2\sqrt{x} + c.$$

$$3. \int \sin 5x \, dx = -\frac{\cos 5x}{5} + c$$

$$4. \int \cos \frac{x}{2} \, dx = \int \cos \frac{1}{2}x \, dx$$

$$= \frac{\sin \left(\frac{1}{2}x\right)}{\frac{1}{2}} + c$$

$$= 2 \sin \frac{x}{2} + c.$$

Rules for Indefinite Integration

1. Common Multiple Rule

$$\int k f(x) \, dx = k \int f(x) \, dx + c \text{ where } k \text{ is constant.}$$

2. Rules of Negatives:

$$\int -f(x) \, dx = - \int f(x) \, dx$$

3. Sum and difference Rule:

$$\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx.$$

Illustrative Examples

Example 1. Evaluate: $\int 3 \sec x \tan x \, dx$

Solution

$$\int 3 \sec x \tan x \, dx = 3 \int \sec x \tan x \, dx.$$

$$= 3 (\sec x + c)$$

$$= 3 \sec x + 3c \text{ (First form)}$$

$$= 3 \sec x + c' \text{ (shorter form where } c' = 3c)$$

$$= 3 \sec x + c. \text{ (Usual form - no prime (dash).)}$$

Since 3 times an arbitrary constant is an arbitrary constant.)

Example 2. (Term by term integration) Evaluate: $\int (x^2 + 3x + 7) \, dx$.

Solution

$$\int (x^2 + 3x + 7) \, dx = \int x^2 \, dx + 3 \int x \, dx + 7 \int 1 \, dx$$

$$= \frac{x^3}{3} + \frac{3x^2}{2} + 7x + c.$$

NOTE We write single arbitrary constant since sum of three arbitrary constants is also an arbitrary constant.

Example 3. Evaluate: $\int \sin^2 x \, dx$

Solution

$$\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx \left[\because \sin^2 A = \frac{1 - \cos 2A}{2} \right]$$

$$= \frac{1}{2} \int (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \int 1 \, dx - \frac{1}{2} \int \cos 2x \, dx$$

$$= \frac{1}{2} x - \frac{1}{2} \cdot \frac{\sin 2x}{2} + c$$

$$= \frac{1}{2} x - \frac{1}{4} \sin 2x + c.$$

Example 4. Evaluate the integral $\int \left(4x^3 - \frac{3}{x} + 5e^x - 2 \right) \, dx$.

Solution

$$\int \left(4x^3 - \frac{3}{x} + 5e^x - 2 \right) \, dx = 4 \int x^3 \, dx - 3 \int \frac{1}{x} \, dx + 5 \int e^x \, dx - 2 \int 1 \, dx$$

$$= 4 \frac{x^{3+1}}{3+1} - 3 \log x + 5e^x - 2x + c$$

$$= x^4 - 3 \log x + 5e^x - 2x + c$$

where c is integral constant.

Example 5. Evaluate the integral $\int \frac{(1-x)^3}{x} \, dx$.

Solution

$$\int \frac{(1-x)^3}{x} \, dx = \int \frac{1 - 3x + 3x^2 - x^3}{x} \, dx$$

$$= \int \left(\frac{1}{x} - 3 + 3x - x^2 \right) \, dx$$

$$= \int \frac{1}{x} \, dx - 3 \int 1 \, dx + 3 \int x \, dx - \int x^2 \, dx$$

$$= \log x - 3x + \frac{3}{2}x^2 - \frac{x^3}{3} + c, \text{ where } c \text{ is an arbitrary constant.}$$

Example 6. Evaluate the following $\int \frac{x-1}{x+1} dx$.

Solution

$$\begin{aligned}\int \frac{x-1}{x+1} dx &= \int \frac{x+1-1-1}{x+1} dx \\ &= \int \frac{x+1}{x+1} dx - 2 \int \frac{1}{x+1} dx \\ &= \int dx - 2 \int \frac{1}{x+1} \\ &= x - 2 \log(x+1) + c, \text{ where } c \text{ is an integral constant.}\end{aligned}$$

Example 7. Evaluate: $\int (\sin 2x - \csc^2 x) dx$.

Solution

$$\begin{aligned}\int (\sin 2x - \csc^2 x) dx &= \int \sin 2x dx - \int \csc^2 x dx \\ &= -\frac{\cos 2x}{2} - (-\cot x) + c \\ &= -\frac{1}{2} \cos 2x + \cot x + c.\end{aligned}$$

Example 8. Evaluate: $\int \frac{dx}{1-\sin x}$.

Solution

$$\begin{aligned}\int \frac{dx}{1-\sin x} \times \frac{1+\sin x}{1+\sin x} &= \int \frac{(1+\sin x) dx}{1-\sin^2 x} \\ &= \int \frac{(1+\sin x) dx}{\cos^2 x} \\ &= \int \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \right) dx \\ &= \int \sec^2 x dx + \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx \\ &= \int \sec^2 x dx + \int \sec x \tan x dx \\ &= \tan x + \sec x + c.\end{aligned}$$

Example 9. Evaluate: $\int \frac{dx}{1-\cos x}$.

Solution

$$\begin{aligned}\int \frac{dx}{1-\cos x} &= \int \frac{dx}{2 \sin^2 \frac{x}{2}} \quad \left[\because 1-\cos A = 2 \sin^2 \frac{A}{2} \right] \\ &= \frac{1}{2} \int \operatorname{cosec}^2 \frac{x}{2} dx \\ &= \frac{1}{2} \left(-\cot \frac{x}{2} \right) + c \\ &= -\cot \frac{x}{2} + c.\end{aligned}$$

Exercise 4.1

1. Evaluate the integrals.

- $\int (x+1) dx$
- $\int (5-6x) dx$
- $\int (ax^2 + bx + c) dx$
- $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3x} \right) dx$
- $\int (\sqrt{x} + \sqrt[3]{x}) dx$
- $\int \left(x + \frac{1}{x} \right)^2 dx$
- $\int \frac{3x^3 + 4x + 5}{x^3} dx$

2. Find the following integrals.

- $\int (7x-2)^3 dx$
- $\int \sqrt{2-5x} dx$
- $\int \frac{1}{\sqrt{2x-9}} dx$
- $\int \frac{dx}{\sqrt{x+1}-\sqrt{x}}$
- $\int \frac{x^2-4}{x+2} dx$
- $\int \frac{2x+1}{x-1} dx$
- $\int \frac{3x+1}{x-2} dx$

3. Evaluate the following integrals.

a. $\int 3 \cos 5\theta \, d\theta$

c. $\int (2 \cos 2x - 3 \sin 3x) \, dx$

e. $\int \frac{1}{\cos^2 x \sin^2 x} \, dx$

g. $\int \sin 4x \cos 2x \, dx$

i. $\int \frac{e^{2x} + e^x + 3}{e^x} \, dx$

b. $\int \frac{\operatorname{cosec} \theta \cot \theta}{2} \, d\theta$

d. $\int \cos^2 x \, dx$

f. $\int \sqrt{1 + \cos x} \, dx$

h. $\int (e^{ax} + e^{-bx}) \, dx$

Very Short Answer Questions

1. Define integration of a function.

2. Write the integral of $\frac{1}{x}$.3. Find the integral of $\sqrt{x}$.4. Find $\int \left(x^2 + \frac{1}{x} + e^x\right) \, dx$.5. Find $\int \sin ax \, dx$.6. Find $\int \sin^2 x \, dx$.**Answers**

1. a. $\frac{x^2}{2} + x + c$

c. $\frac{a^2}{3} + \frac{bx^2}{2} + cx + k$

e. $\frac{2x^{\frac{3}{2}}}{3} + \frac{3x^{\frac{5}{2}}}{4} + c$

g. $3x - \frac{4}{x} - \frac{5}{2x^2} + c$

2. a. $\frac{1}{28}(7x-2)^4 + c$

c. $\sqrt{2x-9} + c$

e. $\frac{x^2}{2} - 2x + c$

g. $3x + 7 \ln(x-2) + c$

b. $5x - 3x^2 + c$

d. $-\frac{1}{x} - \frac{x^3}{3} - \frac{1}{3} \log x + c$

f. $\frac{x^3}{3} + 2x - \frac{1}{x} + c$

b. $-\frac{2}{15}(2-5x)^{\frac{3}{2}} + c$

d. $\frac{2}{3}\{(x+1)^{\frac{3}{2}} + x^{\frac{3}{2}}\} + c$

f. $2x + 3 \ln(x-1) + c$

3. a. $\frac{3}{5} \sin 5\theta + c$

c. $\sin 2x + \cos 3x + c$

e. $\tan x - \cot x + c$

g. $-\frac{\cos 6x}{12} - \frac{\cos 2x}{4} + c$

i. $e^x + x - 3e^{-x} + c$

b. $-\frac{1}{2} \operatorname{cosec} \theta + c$

d. $\frac{1}{2} \left[x + \frac{\sin 2x}{2} \right] + c$

f. $2\sqrt{2} \sin \frac{x}{2} + c$

h. $\frac{e^{ax}}{a} - \frac{e^{-bx}}{b} + c$

Very Short Answer Questions

2. $\ln x + c$

5. $-\frac{\cos ax}{a} + c$

3. $\frac{2}{3} x^{\frac{3}{2}} + c$

6. $\frac{1}{2} x - \frac{\sin 2x}{4} + c$

4. $\frac{x^3}{3} + \ln x + e^x + c$

4.2 Techniques of Integration

There are many functions whose integral can not be found directly by applying the basic integration formulae. The goal of this section is to provide the techniques that change the unfamiliar integrals into the integrals that we can recognize and find their integrals.

There are many techniques of integration. We discuss the following techniques.

- Integration by substitution
 - Integration by parts
 - Integration by partial fractions (simple cases).
- Integration by substitution**

While evaluating an integral, we sometimes substitute a function of integrand and rewrite the integrand in terms of new variable which is easily integrable. We substitute those functions whose derivatives are present in the integrand.

Example 1. Evaluate: $\int 3x^2 (x^3 + 2)^4 \, dx$.

Solution

$$\text{Let } I = \int 3x^2 (x^3 + 2)^4 \, dx$$

$$\text{Put } y = x^3 + 2$$

$$\frac{dy}{dx} = 3x^2$$

$$dy = 3x^2 \, dx$$

Then,

$$I = \int y^4 \, dy$$

$$= \frac{y^5}{5} + c$$

$$= \frac{1}{5} (x^3 + 2)^5 + c.$$

Example 2. Evaluate: $\int \frac{2x+5}{\sqrt{x^2+5x}} dx$.

Solution

$$\text{Let } I = \int \frac{2x+5}{\sqrt{x^2+5x}} dx$$

$$\text{Put } y = x^2 + 5x$$

$$\frac{dy}{dx} = 2x + 5$$

$$\therefore dy = (2x + 5)dx$$

Then,

$$I = \int \frac{dy}{\sqrt{y}}$$

$$= \int \frac{dy}{y^{\frac{1}{2}}}$$

$$= \int y^{-\frac{1}{2}} dy$$

$$= \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{y^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2\sqrt{y} + c$$

$$= 2\sqrt{x^2+5x} + c.$$

Example 3. Evaluate: $\int (2x+3)e^{x^2+3x+5} dx$.

Solution

$$\text{Let } I = \int (2x+3)e^{x^2+3x+5} dx$$

$$\text{Put } y = x^2 + 3x + 5$$

$$dy = (2x+3) dx$$

Then,

$$I = \int e^y dy$$

$$= e^y + c$$

$$= e^{x^2+3x+5} + c.$$

Example 4. $\int \frac{(2 + \frac{1}{x})dx}{2x + \ln x}$

Solution

$$\text{Let } I = \int \frac{(2 + \frac{1}{x})dx}{2x + \ln x}$$

$$\text{Put } y = 2x + \ln x$$

$$\frac{dy}{dx} = 2 + \frac{1}{x}$$

$$\therefore dy = (2 + \frac{1}{x}) dx$$

Then,

$$I = \int \frac{dy}{y}$$

$$= \ln y + c$$

$$= \ln(2x + \ln x) + c.$$

Example 5. Evaluate: $\int (3 \sin x - 4)^2 \cos x dx$.

Solution

$$\text{Let } I = \int (3 \sin x - 4)^2 \cos x dx$$

$$\text{Put } y = 3 \sin x - 4$$

$$\frac{dy}{dx} = 3 \cos x$$

$$dy = 3 \cos x dx$$

$$\frac{dy}{3} = \cos x dx$$

Then,

$$I = \int y^2 \cdot \frac{dy}{3}$$

$$= \frac{1}{3} \int y^2 dy$$

$$= \frac{1}{3} \cdot \frac{y^3}{3} + c$$

$$= \frac{y^3}{9} + c$$

$$= \frac{1}{9} (3 \sin x - 4)^3 + c.$$

Integration by the Method of Trigonometric Substitution

The algebraic expressions of the form $(x^2 + a^2)$, $(a^2 - x^2)$, $(x^2 - a^2)$ can be transformed into a simpler form by the substitution of the trigonometric functions. We note the following substitutions:

1. If the expression is in the form $a^2 - x^2$, then substitute $x = a \sin \theta$ or $x = a \cos \theta$.
2. If the expression is in the form $x^2 + a^2$, then substitute $x = a \tan \theta$ or $x = a \cot \theta$.
3. If the expression is in the form $x^2 - a^2$, then substitute $x = a \sec \theta$ or $x = a \operatorname{cosec} \theta$.

Example 6. Integrate: $\int \frac{1}{(a^2 + x^2)} dx$.

Solution

Let $x = a \tan \theta$. Then $dx = a \sec^2 \theta d\theta$.

Here,

$$\begin{aligned} a^2 + x^2 &= a^2 + a^2 \tan^2 \theta \\ &= a^2 (1 + \tan^2 \theta) \\ &= a^2 \sec^2 \theta. \end{aligned}$$

Now,

$$\begin{aligned} \int \frac{dx}{a^2 + x^2} &= \int \frac{a \sec^2 \theta}{a^2 \sec^2 \theta} d\theta \\ &= \frac{1}{a} \int d\theta \\ &= \frac{1}{a} \theta + c \\ &= \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c. \end{aligned}$$

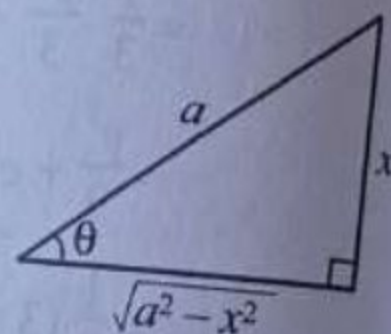
Example 7. Integrate: $\int \frac{a^2}{x^2 \sqrt{a^2 - x^2}} dx$.

Solution

$$\text{Let } I = \int \frac{a^2}{x^2 \sqrt{a^2 - x^2}} dx.$$

Put $x = a \sin \theta$. Then $dx = a \cos \theta d\theta$

and $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = a \cos \theta$.



$$\begin{aligned} I &= \int \frac{a^2 a \cos \theta d\theta}{a^2 \sin^2 \theta a \cos \theta} \\ &= \int \operatorname{cosec}^2 \theta d\theta + c \\ &= \cot \theta + c \\ &= -\frac{\sqrt{a^2 - x^2}}{x} + c \end{aligned}$$

$$\therefore \int \frac{a^2 dx}{x^2 \sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{x} + c.$$

Example 8. Integrate: $\int \frac{1}{x^2 \sqrt{x^2 - a^2}} dx$.

Solution

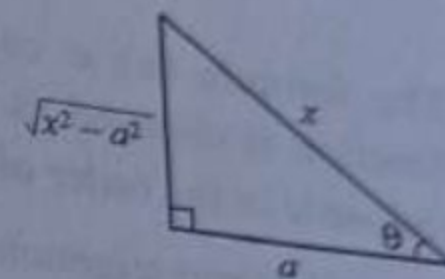
Let $x = a \sec \theta$. Then

$$dx = a \sec \theta \tan \theta d\theta$$

$$\text{and } \sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2} = a \tan \theta.$$

Now,

$$\begin{aligned} \int \frac{2}{x^2 \sqrt{x^2 - a^2}} dx &\text{ becomes } \int \frac{a \sec \theta \tan \theta}{a^2 \sec^2 \theta \cdot a \tan \theta} d\theta \\ &= \frac{1}{a^2} \int \frac{1}{\sec \theta} d\theta \\ &= \frac{1}{a^2} \int \cos \theta d\theta \\ &= \frac{\sin \theta}{a^2} + c \\ &= \frac{\sqrt{x^2 - a^2}}{a^2 x} + c. \end{aligned}$$



b. Integration by parts

Integration by parts is an important technique for the evaluation of integrals. The method is based on the inversion of the formula for differentiation of a product. If u and v are functions of x , then we have,

$$\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}.$$

Integrating both sides with respect to x , we have,

$$\int \frac{d}{dx} (uv) dx = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx$$

$$\text{or, } uv = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx \quad \dots (i)$$

Let $u = f(x)$, and $\frac{dv}{dx} = g(x)$, so that $v = \int g(x) dx$.

$$\text{From (i), } \int f(x) g(x) dx = \int f(x) g(x) dx + \int \left[\int g(x) dx \cdot \frac{d}{dx} (f(x)) \right] dx$$

$$\text{or, } \int f(x) g(x) dx = f(x) \int g(x) dx - \int \left(\frac{d}{dx} f(x) \int g(x) dx \right) dx$$

$$\therefore \int u v dx = u \int v dx - \int \left(\frac{du}{dx} \int v dx \right) dx. \quad \dots (ii)$$

The formula (ii) is called integration by parts formula. The differentiable function is chosen as u and integrable function is chosen as v . We usually choose u in the order of I, L, A, T, E whichever comes first, where

I = Inverse trigonometric function

L = Logarithmic function

A = Algebraic function

T = Trigonometric function

E = Exponential function

Example 9. Evaluate: $\int x e^x dx$.

Solution

$$\int x e^x dx = x \int e^x dx - \int \left(\frac{dx}{dx} \int e^x dx \right) dx$$

$$= x e^x - \int e^x dx$$

$$= x e^x - e^x + c.$$

Example 10. Evaluate: $\int \log x dx$.

Solution

Here,

$$I = \int \log x dx$$

$$= \int 1 \cdot \log x dx$$

$$= \log x \int 1 dx - \int \left[\frac{d}{dx} (\log x) \int 1 dx \right] dx$$

$$= \log x \times x - \int \frac{1}{x} \times x dx$$

$$= x \log x - \int 1 dx$$

$$= x \log x - x + c.$$

Example 11. Evaluate: $\int x^2 \sin x dx$.

Solution

Here,

$$I = \int x^2 \sin x dx$$

$$= x^2 \int \sin x dx - \int \left[\frac{dx^2}{dx} \int \sin x dx \right] dx$$

$$= x^2 (-\cos x) - \int 2x (-\cos x) dx$$

$$= -x^2 \cos x + 2 \int x \cos x dx$$

$$= -x^2 \cos x + 2 \left\{ x \int \cos x dx - \int \left[\frac{dx}{dx} \int \cos x dx \right] dx \right\}$$

$$= -x^2 \cos x + 2 \{ x \sin x - \int 1 \cdot \sin x dx \}$$

$$= -x^2 \cos x + 2x \sin x - 2 \int \sin x dx$$

$$= -x^2 \cos x + 2x \sin x - 2 (-\cos x) + c$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + c.$$

Example 12. Evaluate: $\int e^x \cos x dx$.

Solution

Here,

$$I = \int e^x \cos x dx$$

Integrating by parts, we get

$$I = \cos x \int e^x dx - \int \left[\frac{d}{dx} (\cos x) \int e^x dx \right] dx$$

$$= e^x \cos x - \int (-\sin x) e^x dx$$

$$= e^x \cos x + \int \sin x e^x dx$$

$$= e^x \cos x + \left\{ \sin x \int e^x dx - \int \left[\frac{d}{dx} (\sin x) \int e^x dx \right] dx \right\}$$

$$= e^x \cos x + e^x \sin x - \int \cos x e^x dx$$

$$= e^x \cos x + e^x \sin x - \int e^x \cos x dx$$

$$\text{or, } I = e^x \cos x + e^x \sin x - I \quad [\because I = \int e^x \cos x dx]$$

$$\text{or, } 2I = e^x \cos x + e^x \sin x + c$$

$$\therefore I = \frac{e^x}{2} (\cos x + \sin x) + c.$$

c. Integration by partial fractions

Sometimes, we express the rational fractions into partial fractions. Then we integrate the given function.

	Proper fractions	Partial fractions
1.	$\frac{\text{numerator}}{(x+a)(x+b)}$	$\frac{A}{x+a} + \frac{B}{x+b}$
2.	$\frac{\text{numerator}}{(x+a)^2}$	$\frac{A}{x+a} + \frac{B}{(x+a)^2}$
3.	$\frac{\text{numerator}}{(x^2+a)(x+b)}$	$\frac{Ax+B}{x^2+a} + \frac{C}{x+b}$

Example 13. Evaluate: $\int \frac{2x-11}{x^2+x-2} dx$.

Solution

Let,

$$\frac{2x-11}{x^2+x-2} = \frac{2x-11}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} \quad \dots (i)$$

$$\text{or, } \frac{2x-11}{(x+2)(x-1)} = \frac{A(x-1) + B(x+2)}{(x+2)(x-1)} \quad \dots (ii)$$

$$\text{or, } 2x-11 = A(x-1) + B(x+2)$$

Put $x=1$ in (ii), we get,

$$2-11 = B(1+2)$$

$$B = -3$$

Again, put $x=-2$ in (ii), we get,

$$-4-11 = A(-2-1)$$

$$A = 5$$

Then from (i)

$$\frac{2x-11}{x^2+x-2} = \frac{5}{x+2} - \frac{3}{x-1}$$

Now,

$$\int \frac{2x-11}{x^2+x-2} dx = \int \left(\frac{5}{x+2} - \frac{3}{x-1} \right) dx$$

$$= 5 \int \frac{1}{x+2} dx - 3 \int \frac{1}{x-1} dx$$

$$= 5 \log(x+2) - 3 \log(x-1) + c.$$

Example 14. Evaluate: $\int \frac{6x+7}{(x+2)^2} dx$.

Solution

$$\text{Let } \frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} \quad \dots (i)$$

$$= \frac{A(x+2) + B}{(x+2)^2}$$

$$6x+7 = A(x+2) + B$$

$$\text{Put } x=-2 \text{ in (i), we get,}$$

$$-12+7 = B$$

$$B = -5$$

$$\text{Again, put } x=0 \text{ in (i), we get,}$$

$$6 \times 0 + 7 = 2A + B$$

$$7 = 2A - 5$$

$$2A = 12$$

$$A = 6$$

$$\text{Hence, from (i),}$$

$$\frac{6x+7}{(x+2)^2} = \frac{6}{x+2} - \frac{5}{(x+2)^2}$$

Now,

$$\int \frac{6x+7}{(x+2)^2} dx = \int \frac{6}{x+2} dx - \int \frac{5}{(x+2)^2} dx$$

$$= 6 \int \frac{1}{x+2} dx - 5 \int (x+2)^{-2} dx$$

$$= 6 \cdot \frac{1}{1} \log(x+2) - 5 \frac{(x+2)^{-2+1}}{(-2+1) \cdot 1} + c$$

$$= 6 \log(x+2) + \frac{5}{x+2} + c.$$

Some Standard Integrals

$$1. \int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

$$2. \int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log \frac{a+x}{a-x} + c$$

$$3. \int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log \frac{x-a}{x+a} + c$$

$$4. \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + c$$

$$5. \int \frac{dx}{\sqrt{x^2+a^2}} = \log(x + \sqrt{x^2+a^2}) + c$$

$$6. \int \frac{dx}{\sqrt{x^2-a^2}} = \log(x + \sqrt{x^2-a^2}) + c$$

$$7. \int \sqrt{x^2 + a^2} dx = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \log(x + \sqrt{x^2 + a^2}) + c$$

$$8. \int \sqrt{x^2 - a^2} dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \log(x + \sqrt{x^2 - a^2}) + c$$

$$9. \int \sqrt{a^2 - x^2} dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{1}{2} a^2 \sin^{-1} \frac{x}{a} + c$$

Illustrative Examples

Example 1. Evaluate: $\int \frac{2x-9}{\sqrt{x^2-9x+1}} dx$.

Solution

$$\text{Let } I = \int \frac{2x-9}{\sqrt{x^2-9x+1}} dx$$

$$\text{Put } y = x^2 - 9x + 1$$

$$dy = (2x-9)dx$$

Then,

$$I = \int \frac{dy}{\sqrt{y}}$$

$$= \int \frac{dy}{y^{\frac{1}{2}}}$$

$$= \int y^{-\frac{1}{2}} dy$$

$$= \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

$$= \frac{y^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2\sqrt{y} + c$$

$$= 2\sqrt{x^2-9x+1} + c.$$

Example 2. Evaluate: $\int \sec x dx$.

Solution

$$\text{Let } I = \int \sec x dx$$

$$= \int \frac{\sec x \cdot (\sec x + \tan x)}{\sec x + \tan x} dx$$

$$\text{Put } y = \sec x + \tan x$$

$$\text{Then, } dy = (\sec x \cdot \tan x + \sec^2 x) dx$$

$$dy = \sec x (\sec x + \tan x) dx$$

Then,

$$I = \int \frac{1}{y} dy$$

$$= \log y + c$$

$$= \log (\sec x + \tan x) + c.$$

Example 3. Evaluate: $\int x \cos x dx$.

Solution

$$\int x \cos x dx = x \int \cos x dx - \int \left[\frac{d}{dx}(x) \int \cos x dx \right] dx$$

$$= x \sin x - \int 1 \cdot \sin x dx$$

$$= x \sin x - (-\cos x) + c$$

$$= x \sin x + \cos x + c.$$

Example 4. Evaluate: $\int \frac{dx}{x^2+4}$.

Solution

$$\int \frac{dx}{x^2+4} = \int \frac{dx}{x^2+2^2}$$

$$= \frac{1}{2} \tan^{-1} \frac{x}{2} + c. \quad \left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c \right]$$

Example 5. Evaluate: $\int \frac{dx}{\sqrt{x^2+6x+10}}$.

Solution

$$\int \frac{dx}{\sqrt{x^2+6x+10}} = \int \frac{dx}{\sqrt{x^2+6x+9+1}}$$

$$= \int \frac{dx}{\sqrt{(x+3)^2+1^2}}$$

$$= \log(x+3 + \sqrt{(x+3)^2+1^2}) + c$$

$$= \log(x+3 + \sqrt{x^2+6x+10}) + c.$$

Example 6. Integrate: $\int \frac{dx}{\sqrt{2ax-x^2}}$

Solution

$$\begin{aligned} \text{Let } I &= \int \frac{dx}{\sqrt{2ax-x^2}} \\ &= \int \frac{dx}{\sqrt{a^2-a^2+2ax-x^2}} \\ &= \int \frac{dx}{\sqrt{a^2-(x^2-2ax+a^2)}} \\ &= \int \frac{dx}{\sqrt{a^2-(x-a)^2}} \\ &= \sin^{-1} \left(\frac{x-a}{a} \right) + c. \end{aligned}$$

Example 7. Evaluate: $\int \frac{dx}{e^x + e^{-x}}$

Solution

$$\begin{aligned} \text{Let } I &= \int \frac{dx}{e^x + e^{-x}} \\ &= \int \frac{dx}{e^x + \frac{1}{e^x}} \\ &= \int \frac{e^x dx}{(e^x)^2 + 1} \end{aligned}$$

Put $e^x = y$
 $e^x dx = dy$

$$\therefore I = \int \frac{dy}{y^2 + 1}$$

$$\begin{aligned} &= \frac{1}{1} \tan^{-1} y + c \quad \left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c \right] \\ &= \tan^{-1}(e^x) + c. \end{aligned}$$

Exercise 4.2

1. Evaluate the following integrals (substitution method).

- $\int 2x(x^2-2)^3 dx$
- $\int 2x\sqrt{x^2+1} dx$
- $\int \frac{2x+3}{\sqrt{x^2+3x}} dx$
- $\int (2x+3)e^{x^2+3x+5} dx$
- $\int \frac{(\log x)^5}{x} dx$
- $\int \frac{\sin(\log x)}{x} dx$

2. Evaluate the following integrals (Integration by parts)

- $\int x e^x dx$
- $\int x \ln x dx$
- $\int x \cos x dx$
- $\int x^2 e^x dx$
- $\int e^x \cos x dx$
- $\int (x+1)\sqrt{x-1} dx$

3. Evaluate (trigonometric substitution).

- $\int \frac{dx}{\sqrt{x^2+4}}$
- $\int \frac{x^2 dx}{\sqrt{9-x^2}}$
- $\int \frac{dx}{(x^2-1)^{\frac{3}{2}}}$

4. Evaluate (partial fractions).

- $\int \frac{5x-3}{(x+1)(x-3)} dx$
- $\int \frac{x}{(x-1)(2x+1)} dx$
- $\int \frac{x^2}{(x+2)(x+3)^2} dx$
- $\int \frac{1}{(x-1)(x^2+3)} dx$

5. Evaluate.

- $\int \tan x dx$
- $\int \operatorname{cosec} x dx$
- $\int \frac{dx}{x^2+10x+26}$
- $\int \frac{dx}{\sqrt{x^2-9}}$
- $\int \frac{dx}{\sqrt{x^2+4x+5}}$

Very Short Answer Questions

- Write the formula for $\int u \cdot v \, dx$.
- Evaluate $\int (2ax + b)(ax^2 + bx + c) \, dx$.
- Evaluate $\int \cot x \, dx$.

Answers

- $\frac{(x^2-2)^4}{4} + c$
 - $\frac{2}{3}(x^2+1)^{\frac{3}{2}} + c$
 - $2\sqrt{x^2+3x} + c$
 - $e^{x^2+3x+5} + c$
- $\frac{(\ln x)^3}{6} + c$
 - $-\cos(\ln x) + c$
 - $x \sin x + \cos x + c$
 - $\frac{e^x}{2}(\cos x + \sin x) + c$
- $\frac{2}{3}(x+1)(x-1)^{\frac{1}{2}} - \frac{4}{15}(x-1)^{\frac{5}{2}} + c$
 - $\frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + c$
 - $\frac{1}{3}\log(x-1) + \frac{1}{6}\log(2x+1) + c$
 - $\frac{1}{4}\ln(x-1) - \frac{1}{8}\ln(x^2+3) - \frac{1}{4\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + c$
- $\ln \sec x + c$
 - $\ln(\operatorname{cosec} x - \cot x) + c$
 - $\tan^{-1}(x+5) + c$
 - $\ln(x+2+\sqrt{x^2+4x+5}) + c$
 - $\ln(x+\sqrt{x^2-9}) + c$

Very Short Answer Questions

- $\frac{(ax^2+bx+c)^2}{2} + k$
- $\ln(\sin x) + c$

4.3 Riemann Integrals

In this section, we discuss about Riemann Integrals.

Sigma Notation for Finite Sums

We use the Greek letter Σ (sigma) to write the sum in short form. For instance, the sum $f(t_1)\Delta t + f(t_2)\Delta t + \dots + f(t_n)\Delta t$ can be written as $\sum_{k=1}^n f(t_k)\Delta t$, which is read as 'the sum from k equals 1 to n of f of t_k times delta t '.

The symbol $\sum_{k=1}^n a_k$ denotes the sum $a_1 + a_2 + \dots + a_n$. Here, a_1 is the first term, a_2 is the second term, a_k is the k^{th} term and a_n is the n^{th} (last) term. The variable k is the index of summation. The number 1 is the lower limit of summation and n is the upper limit of the summation.

For example: $\sum_{k=1}^n k^2 = 1^2 + 2^2 + \dots + n^2$.

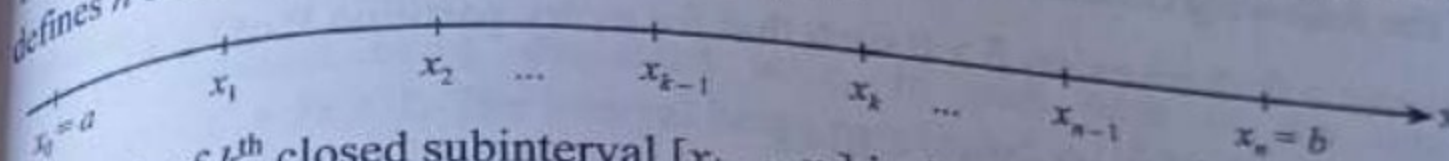
Riemann Sums

Consider an arbitrary continuous function $y = f(x)$ on an interval $[a, b]$. We partition the interval into n subintervals by choosing $n-1$ points. Let these points be $x_1, x_2, \dots, x_{n-1}$ such that

$$a < x_1 < x_2 < \dots < x_{n-1} < b.$$

We usually denote a by x_0 and b by x_n .

The set $P = \{x_0, x_1, \dots, x_n\}$ is called a partition of $[a, b]$. The partition P defines n closed subintervals $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$.



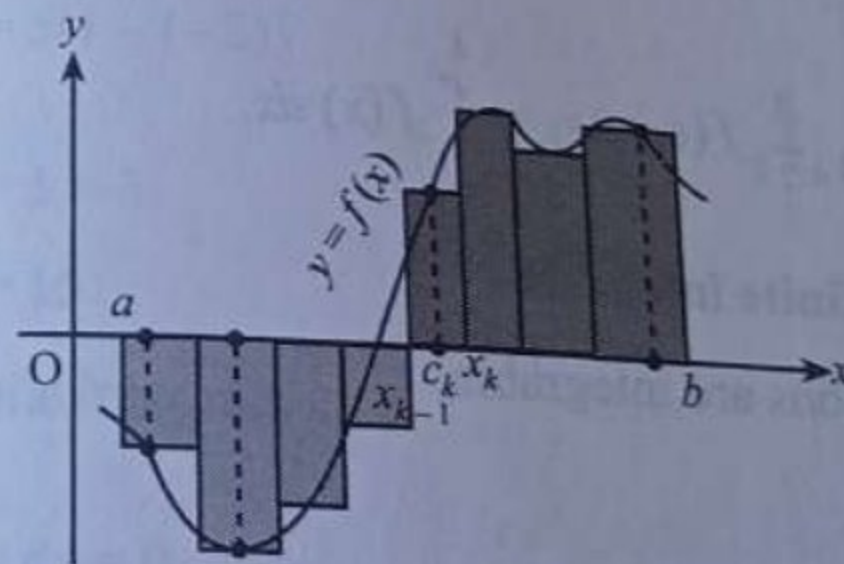
The length of k^{th} closed subinterval $[x_{k-1}, x_k]$ is denoted by

$$\Delta x_k = x_k - x_{k-1}$$

In each subinterval $[x_{k-1}, x_k]$, we select a point c_k and construct a vertical rectangle from the subinterval to the point $(c_k, f(c_k))$ on the curve $y = f(x)$.

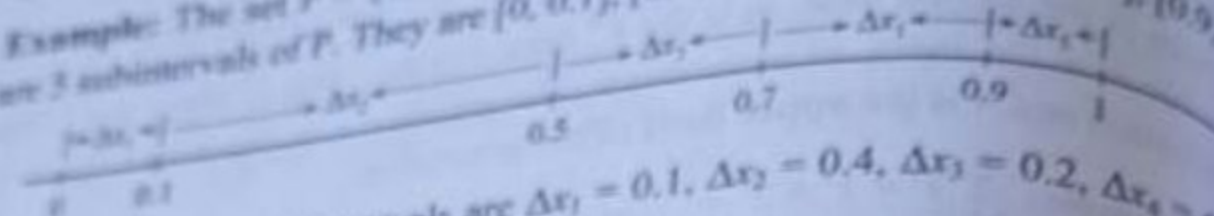
If $f(c_k)$ is positive, the number $f(c_k) \Delta x_k = \text{height} \times \text{base}$, is the area of the rectangle. If $f(c_k)$ is negative, then $f(c_k) \Delta x_k$ is the negative of the area of rectangle. In any case, let $s_p = \sum_{k=1}^n f(c_k) \Delta x_k$.

This sum, which depends on P and the choice of the number c_k , is called a Riemann sum for f on the interval $[a, b]$.



The norm of a partition P is the partition's longest subinterval length. It is denoted by $\|P\|$ read as 'norm of P '.

Example: The set $P = \{0, 0.1, 0.5, 0.7, 0.9, 1\}$ is a partition of $[0, 1]$. There are 5 subintervals of P . They are $[0, 0.1]$, $[0.1, 0.5]$, $[0.5, 0.7]$, $[0.7, 0.9]$, $[0.9, 1]$.



The length of the sub-intervals are $\Delta x_1 = 0.1$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.2$, $\Delta x_4 = 0.2$, $\Delta x_5 = 0.1$.

The longest sub-interval length is 0.4.

$$\therefore \|P\| = 0.4$$

The definite integral as a limit of Riemann Sums

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that the limit of the Riemann sums $\sum_{k=1}^n f(c_k) \Delta x_k$ on $[a, b]$ as $\|P\| \rightarrow 0$ is the number I if the following condition is satisfied.

Given $\epsilon > 0$, there exists $\delta > 0$ such that for every partition P of $[a, b]$

$$\|P\| < \delta \Rightarrow \left| \sum_{k=1}^n f(c_k) \Delta x_k - I \right| < \epsilon$$

for any choice of the number c_k in the subintervals $[x_{k-1}, x_k]$. If the limit exists, we write,

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k = I$$

We call I the definite integral of f over $[a, b]$. Here, we say that f is Riemann integrable ($\mathbb{R}$ -integrable) over $[a, b]$, and we say that the Riemann sums of f on $[a, b]$ converge to the number I .

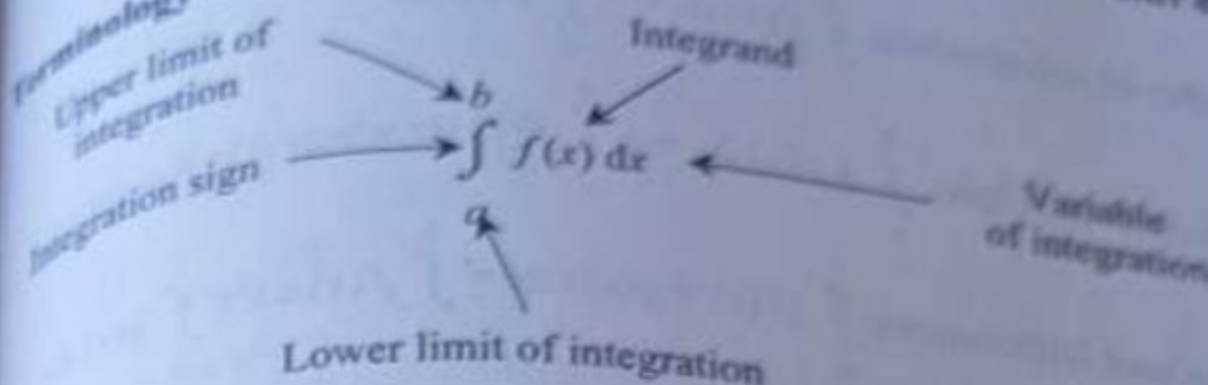
We write I as $\int_a^b f(x) dx$, which is read 'integral of f from a to b '. Thus, if the

$$\text{limit exists, } \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k = \int_a^b f(x) dx.$$

The existence of definite integrals

All continuous functions are integrable.

Terminology for integral of f from a to b



NOTE

The variable of integration is dummy variable in definite integral.

For example, we can use $\int_a^b f(t) dt$ or $\int_a^b f(u) du$ instead of $\int_a^b f(x) dx$.

Example 1. Express the limit of Riemann sums $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (c_k^2 + 3c_k + 1) \Delta x_k$ as an integral if P denotes a partition of the interval $[1, 2]$.

Solution

The function being evaluated at c_k in each term of the sum is $f(x) = x^2 + 3x + 1$. The interval being partitioned is $[1, 2]$. Hence, the limit of the integral of f is from 1 to 2.

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (c_k^2 + 3c_k + 1) \Delta x_k = \int_1^2 (x^2 + 3x + 1) dx.$$

NOTE

If $f(x)$ has the constant value c on $[a, b]$, then $\int_a^b f(x) dx = \int_a^b c dx = c(b-a)$.

Example 2. $\int_{-2}^1 5 dx$.

Solution

$$\begin{aligned} \int_{-2}^1 5 dx &= 5(1 - (-2)) \\ &= 5 \times 3 \\ &= 15. \end{aligned}$$

Properties of Definite Integrals

$$1. \text{ Zero: } \int_a^a f(x) dx = 0$$

2. **Order of integration:** $\int_a^b f(x) dx = -\int_b^a f(x) dx$
3. **Constant multiple:** $\int_a^b k f(x) dx = k \int_a^b f(x) dx$ where k is a constant.
4. **Sums and Differences:** $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
5. **Additivity:** $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$
6. **Max - Min Inequality:** If $\max f$ and $\min f$ are maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b-a) \leq \int_a^b f(x) dx \leq \max f \cdot (b-a)$$
7. **Domination:** $f(x) \geq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x) dx \geq \int_a^b g(x) dx$
 Also, $f(x) \geq 0$ on $[a, b] \Rightarrow \int_a^b f(x) dx \geq 0$.

4.4 The Fundamental Theorem

The fundamental theorem of calculus is the independent discovery by Leibniz and Newton. It gives the connection between differentiation and integration.

Theorem (The Fundamental Theorem of Calculus, Part 1)

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t) dt$ has a derivative at every

point of $[a, b]$ and $\frac{dF}{dx} = \frac{d}{dx} \int_a^x f(t) dt = f(x); a \leq x \leq b$.

Example 3. Find $\frac{dy}{dx}$ if $y = \int_1^{x^2} \cos t dt$.

Solution

Here, the upper limit of integration is not x but x^2 . To find $\frac{dy}{dx}$, we treat y as the composite of $y = \int_1^u \cos t dt$ and $u = x^2$

By chain rule, we have,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{d}{du} \left(\int_1^u \cos t dt \right) \cdot \frac{du}{dx} \\ &= \cos u \cdot \frac{du}{dx} \\ &= \cos x^2 \cdot 2x \\ &= 2x \cos x^2. \end{aligned}$$

Theorem (The Fundamental Theorem of Calculus, Part 2).

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then $\int_a^b f(x) dx = F(b) - F(a)$.

Steps of finding $\int_a^b f(x) dx$

1. Find an antiderivative F of f .
2. Calculate $F(b) - F(a)$.
3. The number obtained in step - 2 is the value of $\int_a^b f(x) dx$.

Example 4. Evaluate: $\int_{-2}^0 (2x + 5) dx$.

Solution

$$\begin{aligned} \int_{-2}^0 (2x + 5) dx &= \left[2 \cdot \frac{x^2}{2} + 5x \right]_{-2}^0 \\ &= [x^2 + 5x]_{-2}^0 \\ &= (0^2 + 5 \times 0) - \{(-2)^2 + 5 \times (-2)\} \\ &= 0 - (4 - 10) \\ &= 0 + 6 \\ &= 6. \end{aligned}$$

Illustrative Examples

Example 1. Evaluate: $\int_0^1 (x^2 + 5) dx$.

Solution

$$\begin{aligned}\int_0^1 (x^2 + 5) dx &= \left[\frac{x^3}{3} + 5x \right]_0^1 \\ &= \left(\frac{1^3}{3} + 5 \times 1 \right) - \left(\frac{0^3}{3} + 5 \times 0 \right) \\ &= \frac{1}{3} + 5 = \frac{1+15}{3} \\ &= \frac{16}{3}.\end{aligned}$$

Example 2. Evaluate: $\int_0^\pi \cos x dx$.

Solution

$$\begin{aligned}\int_0^\pi \cos x dx &= [\sin x]_0^\pi \\ &= \sin \pi - \sin 0 \\ &= 0 - 0 \\ &= 0.\end{aligned}$$

Example 3. Find the value of $\int_{-\pi/3}^{\pi/3} \cos t dt$.

Solution

Here,

$$\begin{aligned}I &= \int_{-\pi/3}^{\pi/3} \cos t dt = [\sin t]_{-\pi/3}^{\pi/3} \\ &= \sin \frac{\pi}{3} - \sin \left(-\frac{\pi}{3} \right) \\ &= \sin \frac{\pi}{3} + \sin \frac{\pi}{3} \quad [\because \sin(-\theta) = -\sin \theta] \\ &= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \\ &= \frac{2\sqrt{3}}{2} \\ &= \sqrt{3}.\end{aligned}$$

Example 4. Evaluate: $\int_0^2 \frac{x dx}{\sqrt{x^2 + 4}}$.

Solution

$$\text{We have, } I = \int_0^2 \frac{x dx}{\sqrt{x^2 + 4}}$$

$$\text{Put, } x^2 + 4 = y$$

$$\text{Then, } 2x = \frac{dy}{dx}$$

$$x dx = \frac{1}{2} dy$$

$$\text{When } x = 0, y = 0^2 + 4 = 4$$

$$\text{When } x = 2, y = 2^2 + 4 = 8$$

Now,

$$\begin{aligned}\int_0^2 \frac{x dx}{\sqrt{x^2 + 4}} &= \int_4^8 \frac{1}{2\sqrt{y}} dy \\ &= \frac{1}{2} \int_4^8 \frac{dy}{\sqrt{y}} \\ &= \frac{1}{2} \int_4^8 y^{-1/2} dy \\ &= \frac{1}{2} \left[\frac{y^{-1/2+1}}{-1/2+1} \right]_4^8 \\ &= \frac{1}{2} \left[\frac{y^{1/2}}{1/2} \right]_4^8 \\ &= \frac{1}{2} \times 2 \left[y^{1/2} \right]_4^8 \\ &= 8^{1/2} - 4^{1/2} \\ &= \sqrt{8} - \sqrt{4} \\ &= 2\sqrt{2} - 2 \\ &= 2(\sqrt{2} - 1).\end{aligned}$$

$$\text{Hence, } \int_0^2 \frac{x dx}{\sqrt{x^2 + 4}} = 2(\sqrt{2} - 1).$$

Example 5. Evaluate: $\int_0^1 x e^x dx$.

Solution

$$\begin{aligned} \int_0^1 x e^x dx &= \int x e^x dx \\ \text{Indefinite Integral} &= \int x e^x dx \\ &= x \int e^x dx - \int \left\{ \frac{d}{dx}(x) \int e^x dx \right\} dx \\ &= x e^x - \int 1 \cdot e^x dx \\ &= x e^x - e^x \\ \text{Definite Integral} &= [x e^x - e^x]_0^1 \\ &= (1 \cdot e^1 - e^1) - (0 \cdot e^0 - e^0) \\ &= 0 - (0 - 1) \\ &= 1. \end{aligned}$$

Example 6. Evaluate: $\int_0^{\pi/2} \frac{dx}{1 + \sin x}$.

Solution

$$\begin{aligned} \int_0^{\pi/2} \frac{dx}{1 + \sin x} &= \int_0^{\pi/2} \frac{dx}{1 - \cos\left(\frac{\pi}{2} + x\right)} \quad \left[\because \cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta \right] \\ &= \int_0^{\pi/2} \frac{dx}{2 \sin^2 \frac{1}{2}\left(\frac{\pi}{2} + x\right)} \quad \left[\because 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2} \right] \\ &= \frac{1}{2} \int_0^{\pi/2} \operatorname{cosec}^2 \frac{1}{2}\left(\frac{\pi}{2} + x\right) dx = \frac{1}{2} \left[\frac{-\cot \frac{1}{2}\left(\frac{\pi}{2} + x\right)}{\frac{1}{2}} \right]_0^{\pi/2} \\ &= \left[-\cot\left(\frac{\pi}{4} + \frac{x}{2}\right) \right]_0^{\pi/2} = -\left[\cot\left(\frac{\pi}{4} + \frac{\pi}{4}\right) - \cot \frac{\pi}{4} \right] \\ &= -\cot \frac{\pi}{2} + \cot \frac{\pi}{4} \\ &= -0 + 1 \\ &= 1. \end{aligned}$$

Exercise 4.3

Evaluate the following integrals.

- a. $\int_0^2 (x+4) dx$
- b. $\int_0^1 (2x+3) dx$
- c. $\int_1^3 (2x^2+5x) dx$
- d. $\int_1^4 (3x^2+2x+5) dx$
- e. $\int_0^2 \left(x^3 + \frac{x^2}{2} + x\right) dx$
- f. $\int_{-2}^2 (x^3 - 2x + 3) dx$

Evaluate the following integrals.

- a. $\int_0^{\pi} \sin x dx$
- b. $\int_0^{\pi} (1 + \cos x) dx$
- c. $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc \theta \cot \theta d\theta$
- d. $\int_{\frac{\pi}{2}}^0 \frac{1 + \cos 2\theta}{2} d\theta$
- e. $\int_0^{\pi} \sin^2 \left(1 + \frac{\theta}{2}\right) d\theta$

3. Find $\frac{dy}{dx}$ if $y = \int_0^{\sqrt{x}} \cos t dt$

4. Evaluate the following definite integrals.

- a. $\int_0^1 \frac{2x dx}{x^2+1}$
- b. $\int_0^1 \frac{16x dx}{8x^2+2}$
- c. $\int_0^{\frac{\pi}{2}} x \sin x dx$
- d. $\int_2^1 \frac{\sin(\log t)}{t} dt$
- e. $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$

Very Short Answer Questions

- The set $P = \{0, 0.1, 0.2, 0.7, 0.9, 1\}$ is a partition of $[0, 1]$. Then find $|P|$.
- If $f(x)$ has the constant value c on $[a, b]$ then find $\int_a^b f(x) dx$.
- Evaluate $\int_1^{\sqrt{e}} x dx$.
- Evaluate $\int_0^{\pi} \sin x dx$.
- Evaluate $\int_{-\pi}^{\pi} \sin x dx$.
- What do you mean by definite integral?
- If $\frac{d}{dx}[f(x)] = F(x)$ then find $\int_a^b F(x) dx$.
- Write the meaning of $\int_a^b f(x) dx$.

Answers

- | | | | |
|--|-------------|--------------------|----------------------|
| 1. a. 10 | b. 4 | c. $\frac{112}{3}$ | d. 93 |
| e. $\frac{22}{3}$ | f. 12 | | |
| 2. a. 2 | b. π | c. 0 | d. $-\frac{\pi}{4}$ |
| e. $\frac{\pi}{2} + \sin 2$ | | | |
| 3. $\frac{1}{2\sqrt{x}} \cos \sqrt{x}$ | | | |
| 4. a. $\log 2$ | b. $\log 5$ | c. 1 | d. $\cos \log 2 - 1$ |
| e. $\frac{\pi}{6}$ | | | |

Very Short Answer Questions

- | | | | |
|--------|------------------|------|------|
| 1. 0.5 | 2. $c(b-a)$ | 3. 2 | 4. 2 |
| 5. 0 | 7. $f(b) - f(a)$ | | |

4.5 Improper Integrals

Up to now, we have required the definite integrals to have two properties. They are

- The domain of integration, say, from a to b , is finite.
 - The range of the integrand be finite on this domain.
- When one or both of these conditions are not satisfied, we do the finite case and then see what happened as it approaches infinity.

Definition

An integral is an improper integral if either the interval of integration is not finite improper integral of type 1 or if the function to integrate is not continuous (not bounded) in the interval of integration (improper integral of type 2).

Examples:

- $\int_0^{\infty} e^{-x} dx$ is an improper integral of type 1 since the upper limit of integration is infinite.
- $\int_0^1 \frac{dx}{x}$ is an improper integral of type 2 because $\frac{1}{x}$ is not continuous at 0.

Thus, integrals with infinite limits of integration and integrals of function that become infinite at a point within the interval of integration are improper integrals. When the limits involved exist, we evaluate such integrals with the following definitions:

- If f is continuous on $[a, \infty)$, then $\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$.
- If f is continuous on $(-\infty, b]$, then $\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$.
- If f is continuous on $(a, b]$, then $\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx$.
- If f is continuous on $[a, b)$, then $\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx$.

In each case, if the limit is finite, we say that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist the improper integral **diverges**.

Example 1. Evaluate: $\int_1^{\infty} \frac{dx}{x^2}$

Solution

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x^2} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^2} \quad [\text{Infinite upper limit of integration}] \\ &= \lim_{b \rightarrow \infty} \int_1^b x^{-2} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{x^{-2+1}}{-2+1} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left[\frac{-1}{x} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left[\frac{-1}{b} - \left(\frac{-1}{1} \right) \right] \\ &= \lim_{b \rightarrow \infty} \left(1 - \frac{1}{b} \right) \\ &= 1 - 0 \\ &= 1.\end{aligned}$$

Definition

If f becomes infinite at an interior point d of $[a, b]$, then

$$\int_a^b f(x) dx = \int_a^d f(x) dx + \int_d^b f(x) dx$$

The integral from a to b **converges** if the integrals from a to d and d to b both converge. Otherwise, the integral from a to b **diverges**.

Example 2. Investigate the convergence of $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$

Solution

The integrand $f(x) = \frac{1}{(x-1)^{\frac{2}{3}}}$ becomes infinite at $x = 1$ but it is continuous on $[0, 1)$ and $(1, 3]$. The convergence of the integral over $[0, 3]$ depends on the integrals from 0 to 1 and 1 to 3.
On $[0, 1]$, we have

$$\begin{aligned}\int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} &= \lim_{b \rightarrow 1^-} \int_0^b (x-1)^{-\frac{2}{3}} dx \\ &= \lim_{b \rightarrow 1^-} \left[\frac{(x-1)^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} \right]_0^b \\ &= \lim_{b \rightarrow 1^-} \left[3(x-1)^{\frac{1}{3}} \right]_0^b \\ &= \lim_{b \rightarrow 1^-} [3(b-1)^{\frac{1}{3}} - 3(0-1)^{\frac{1}{3}}] \\ &= 3.\end{aligned}$$

On $[1, 3]$, we have,

$$\begin{aligned}\int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}} &= \lim_{c \rightarrow 1^+} \int_c^3 \frac{dx}{(x-1)^{\frac{2}{3}}} \\ &= \lim_{c \rightarrow 1^+} [3(x-1)^{\frac{1}{3}}]_c^3 \\ &= \lim_{c \rightarrow 1^+} [3(3-1)^{\frac{1}{3}} - (c-1)^{\frac{1}{3}}] \\ &= 3 \cdot 2^{\frac{1}{3}} \\ &= 3\sqrt[3]{2}.\end{aligned}$$

Both the limits are finite, so the integral of f from 0 to 3 converges and its value is $3 + 3\sqrt[3]{2}$.

Definition (Integrals from $-\infty$ to ∞)

If f is continuous on $(-\infty, \infty)$ and if $\int_{-\infty}^a f(x) dx$ and $\int_a^{\infty} f(x) dx$ both converge then we say that $\int_{-\infty}^{\infty} f(x) dx$ converges and define its value to be

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx.$$

If either or both of the integrals on the right hand side of this equation diverge then the integral of f from $-\infty$ to ∞ diverges.

Example 1. Evaluate: $\int_1^{\infty} \frac{dx}{x^2}$

Solution

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^2} \quad (\text{Infinite upper limit of integration})$$

$$= \lim_{b \rightarrow \infty} \int_1^b x^{-2} dx$$

$$= \lim_{b \rightarrow \infty} \left[\frac{x^{-2+1}}{-2+1} \right]_1^b$$

$$= \lim_{b \rightarrow \infty} \left[\frac{-1}{x} \right]_1^b$$

$$= \lim_{b \rightarrow \infty} \left[\frac{-1}{b} - \left(\frac{-1}{1} \right) \right]$$

$$= \lim_{b \rightarrow \infty} \left(1 - \frac{1}{b} \right)$$

$$= 1 - 0$$

$$= 1.$$

Definition

If f becomes infinite at an interior point d of $[a, b]$, then

$$\int_a^b f(x) dx = \int_a^d f(x) dx + \int_d^b f(x) dx$$

The integral from a to b **converges** if the integrals from a to d and d to b both converge. Otherwise, the integral from a to b **diverges**.

Example 2. Investigate the convergence of $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$

Solution

The integrand $f(x) = \frac{1}{(x-1)^{\frac{2}{3}}}$ becomes infinite at $x = 1$ but it is continuous

on $[0, 1)$ and $(1, 3]$. The convergence of the integral over $[0, 3]$ depends on the integrals from 0 to 1 and 1 to 3.

On $[0, 1]$, we have

$$\int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{b \rightarrow 1^-} \int_0^b (x-1)^{-\frac{2}{3}} dx$$

$$= \lim_{b \rightarrow 1^-} \left[\frac{(x-1)^{\frac{1}{3}}}{\frac{1}{3}} \right]_0^b$$

$$= \lim_{b \rightarrow 1^-} \left[3(x-1)^{\frac{1}{3}} \right]_0^b$$

$$= \lim_{b \rightarrow 1^-} [3(b-1)^{\frac{1}{3}} - 3(0-1)^{\frac{1}{3}}]$$

$$= 3.$$

On $[1, 3]$, we have,

$$\int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{c \rightarrow 1^+} \int_c^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$$

$$= \lim_{c \rightarrow 1^+} [3(x-1)^{\frac{1}{3}}]_c^3$$

$$= \lim_{c \rightarrow 1^+} [3(3-1)^{\frac{1}{3}} - (c-1)^{\frac{1}{3}}]$$

$$= 3 \cdot 2^{\frac{1}{3}}$$

$$= 3\sqrt[3]{2}.$$

Both the limits are finite, so the integral of f from 0 to 3 converges and its value is $3 + 3\sqrt[3]{2}$.

Definition (Integrals from $-\infty$ to ∞)

If f is continuous on $(-\infty, \infty)$ and if $\int_{-\infty}^a f(x) dx$ and $\int_a^{\infty} f(x) dx$ both converge

then we say that $\int_{-\infty}^{\infty} f(x) dx$ converges and define its value to be

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx.$$

If either or both of the integrals on the right hand side of this equation diverge then the integral of f from $-\infty$ to ∞ diverges.

Example 3. $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2}$

Solution

$$\begin{aligned} & \lim_{b \rightarrow -\infty} [\tan^{-1} x]_b^0 + \lim_{c \rightarrow \infty} [\tan^{-1} x]_0^c \\ &= \lim_{b \rightarrow -\infty} [\tan^{-1} 0 - \tan^{-1} b] + \lim_{c \rightarrow \infty} [\tan^{-1} c - \tan^{-1} 0] \\ &= 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0 \\ &= \pi. \end{aligned}$$

The integral $\int_1^{\infty} \frac{dx}{x^p}$

The integral $\int_1^{\infty} \frac{dx}{x^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Example 4. Investigate the convergence of $\int_1^{\infty} \frac{dx}{x}$ and $\int_1^{\infty} \frac{dx}{x^2}$.

Solution

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x} \\ &= \lim_{b \rightarrow \infty} (\ln b - \ln 1) \\ &= \infty \end{aligned}$$

Hence, the integral diverges:

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x^2} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^2} \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{x}\right]_1^b \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + 1\right) \\ &= 1 \end{aligned}$$

So the integral converges, and the value of integral is 1.

Tests for Convergence and Divergence

Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ and suppose that $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\int_a^{\infty} f(x) dx$ converges if $\int_a^{\infty} g(x) dx$ converges.
2. $\int_a^{\infty} g(x) dx$ diverges if $\int_a^{\infty} f(x) dx$ diverges.

Examples

1. $\int_1^{\infty} \frac{\sin^2 x}{x^2} dx$ converges because $0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ on $[1, \infty)$ and $\int_1^{\infty} \frac{1}{x^2} dx$ converges.
2. $\int_1^{\infty} \frac{1}{\sqrt{x^2 - 0.1}} dx$ diverges because $\frac{1}{\sqrt{x^2 - 0.1}} \geq \frac{1}{x}$ on $[1, \infty)$ and $\int_1^{\infty} \frac{1}{x} dx$ diverges.

Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$ and if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$ ($0 < L < \infty$), then $\int_a^{\infty} f(x) dx$ and $\int_a^{\infty} g(x) dx$ both converge or both diverge.

Illustrative Examples

Example 1. Evaluate: $\int_0^1 \frac{1}{\sqrt{x}} dx$.

Solution

$$\begin{aligned} \int_0^1 \frac{1}{\sqrt{x}} dx &= \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{\sqrt{x}} dx = \lim_{a \rightarrow 0^+} \int_a^1 x^{-\frac{1}{2}} dx \\ &= \lim_{a \rightarrow 0^+} \left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right]_a^1 = \lim_{a \rightarrow 0^+} [2\sqrt{x}]_a^1 \\ &= \lim_{a \rightarrow 0^+} [2 - 2\sqrt{a}] \\ &= 2. \end{aligned}$$

Example 2. Investigate the convergence of $\int_0^1 \frac{1}{1-x} dx$.

Solution

The integrand $f(x) = \frac{1}{1-x}$ is continuous on $[0, 1]$ but becomes infinite as $x \rightarrow 1^-$.

Now,

$$\begin{aligned}\int_0^1 \frac{1}{1-x} dx &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{1-x} dx \\ &= \lim_{b \rightarrow 1^-} [-\ln|1-x|]_0^b \\ &= \lim_{b \rightarrow 1^-} [-\ln(1-b) + 0] \\ &= \infty.\end{aligned}$$

The limit is infinite, so the integral diverges.

Exercise 4.4

1. Evaluate the following improper integrals

a. $\int_0^{\infty} \frac{dx}{x^2+1}$

b. $\int_0^1 \frac{dx}{\sqrt{x}}$

c. $\int_{-1}^1 \frac{dx}{x^2}$

d. $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

e. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

2. Test for convergence and divergence for the following integrals.

a. $\int_1^{\infty} \frac{dx}{x}$

b. $\int_0^{\frac{\pi}{2}} \tan \theta d\theta$

Very Short Answer Questions

What do you mean by improper integral?

1. Evaluate $\int_1^{\infty} \frac{dx}{x^2}$.

2. Evaluate $\int_0^{\infty} \frac{dx}{1+x^2}$.

3. Evaluate $\int_1^{\infty} \frac{dx}{x^3}$.

4. When does the integral $\int_1^{\infty} \frac{dx}{x^p}$ converge?

Answers

1. a. $\frac{\pi}{2}$ b. 2 c. 6 d. π

2. a. Diverges b. Diverges

Very Short Answer Questions

2. 1 3. $\frac{\pi}{2}$ 4. $\frac{1}{2}$

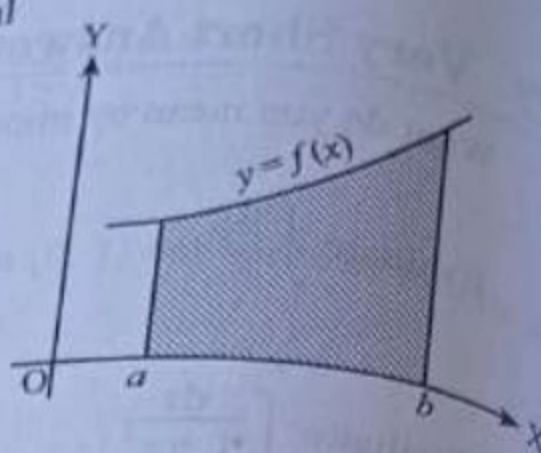
4.6 Applications of Definite Integrals

The Definite integral was initially developed to determine the areas of regions enclosed by curves of different sizes and shapes without the reference to indefinite integral. Later, with the help of fundamental theorem of integral calculus, the definite integrals can be evaluated through indefinite integrals.

With the help of integrals we can calculate areas between the curves, the volumes and surface areas of solids, the length of curves, etc. We define all of these as limits of Riemann sums of continuous functions on closed intervals, that is, as integrals, and evaluate these limits with calculus.

Theorem: If f is continuous in the interval (a, b) then $\int_a^b f(x) dx$ exists.

Formula 1. If $f(x)$ is continuous on $[a, b]$ then the area bounded by the curve $y = f(x)$, the x -axis and the two ordinates $x = a$, $x = b$ is equal to $\int_a^b f(x) dx$ or $\int_a^b y dx$.



Example 1. Find the area bounded by the x -axis and the curve $y = 4x^3$ and the ordinates at $x = 2$ and $x = 4$.

Solution

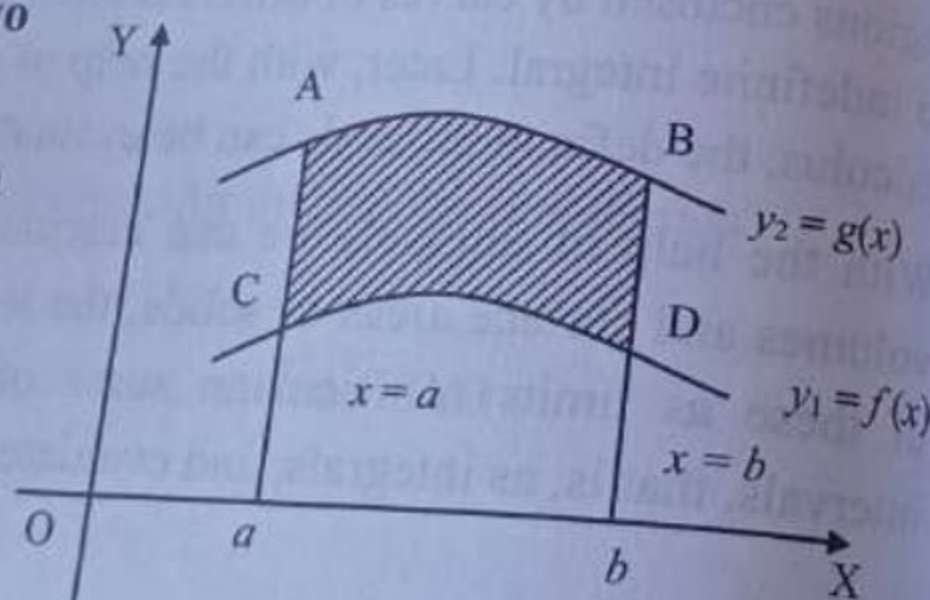
Here, $y = 4x^3$, $x = 2$ and $x = 4$.

$$\begin{aligned} \therefore \text{The required area } A &= \int_2^4 y dx \\ &= \int_2^4 4x^3 dx \\ &= 4 \left[\frac{x^4}{4} \right]_2^4 \\ &= \frac{4}{4} [4^4 - 2^4] \\ &= [256 - 16] \\ &= 240 \text{ sq. units.} \end{aligned}$$

Formula 2. The area bounded by the curve $x = f(y)$, the y -axis and the two abscissae $y = a$ and $y = b$ is equal to $\int_a^b f(y) dy$ or $\int_a^b x dy$.

Formula 3 (Area Between Two curves)

Here, we want to find the area between the curves $y_1 = f(x)$ and $y_2 = g(x)$, and the lines $x = a$ and $x = b$. The area is the difference between the areas under the two curves.



Thus,
$$A = \int_a^b y_2 dx - \int_a^b y_1 dx = \int_a^b (y_2 - y_1) dx.$$

Hence, the area of the region bounded by the curves $y_1 = f(x)$, $y_2 = g(x)$, and between the lines $x = a$ and $x = b$ is given by

$$A = \int_a^b (y_2 - y_1) dx = \int_a^b [g(x) - f(x)] dx.$$

Example 2. Find the area bounded by the curve $y^2 = 4x$ and the line $y = x$.

Solution

Here given curve and line are

$$y^2 = 4x \quad \dots (i)$$

$$\text{and } y = x \quad \dots (ii)$$

Solving (i) and (ii), we get,

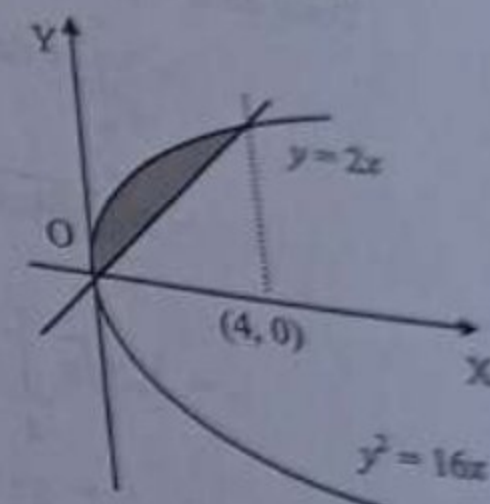
$$x^2 = 4x$$

$$x^2 - 4x = 0$$

$$\Rightarrow x(x - 4) = 0$$

$$\text{i.e., } x = 0 \text{ and } x = 4$$

i.e., $x = 0$ and $x = 4$ are the ordinates of the points at which the given curves intersect.



$$\begin{aligned} \therefore \text{The required area} &= \int_0^4 (y_1 - y_2) dx \\ &= \int_0^4 (\sqrt{4x} - x) dx \\ &= 2 \int_0^4 \sqrt{x} dx - \int_0^4 x dx \\ &= 2 \cdot \frac{2}{3} [x^{3/2}]_0^4 - \frac{1}{2} [x^2]_0^4 \\ &= \frac{4}{3} [4^{3/2} - 0] - \frac{1}{2} [4^2 - 0^2] \\ &= \frac{32}{3} - 8 \\ &= \frac{8}{3} \text{ sq. units.} \end{aligned}$$

4.7 Quadrate, Rectification, Volume and Surface Integral

Quadrate

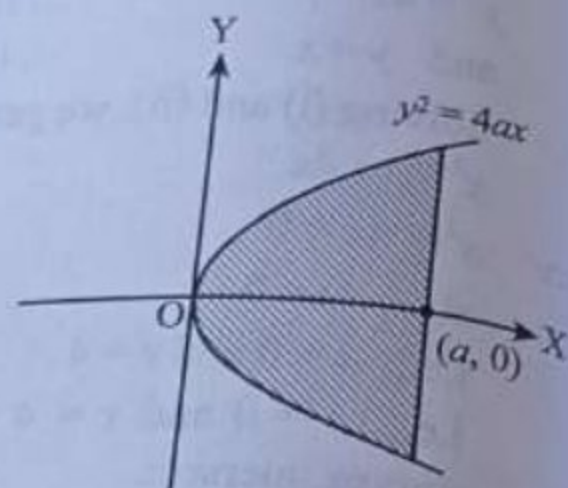
Quadrate means being square or approximately square. We draw a square with the same area as a given plane figure. It helps us to find the area bounded by the curves.

Example 3. Find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum.

Solution

Since the curve (parabola) is symmetric about x-axis, so
Total area = 2 (area of the region in first quadrant)

$$\begin{aligned} &= 2 \int_0^a y \, dx \\ &= 2 \int_0^a \sqrt{4ax} \, dx \\ &= 2 \cdot 2\sqrt{a} \int_0^a x^{\frac{1}{2}} \, dx \\ &= 4\sqrt{a} \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^a \\ &= \frac{8\sqrt{a}}{3} [x^{\frac{3}{2}}]_0^a \\ &= \frac{8\sqrt{a}}{3} [a^{\frac{3}{2}} - 0] \\ &= \frac{8a^2}{3} \end{aligned}$$



Rectification

Rectification means the process of finding a straight line equal in length to a given arc.

Definition: A function with a continuous first derivative is said to be **smooth** and its graph is called a **smooth curve**.

Definition: If f is smooth on $[a, b]$, the **length** of the curve $y = f(x)$ from a to b is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx.$$

Example 4. Find the length of the curve $y = \frac{4\sqrt{2}}{3} x^{\frac{3}{2}} - 1, 0 \leq x \leq 1$.

$$\begin{aligned} \text{Here, } a &= 0, b = 1 \\ \text{And, } y &= \frac{4\sqrt{2}}{3} x^{\frac{3}{2}} - 1 \\ \frac{dy}{dx} &= \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{\frac{1}{2}} - 0 \\ &= 2\sqrt{2} x^{\frac{1}{2}} \\ \left(\frac{dy}{dx}\right)^2 &= (2\sqrt{2} x^{\frac{1}{2}})^2 \\ &= 8x. \end{aligned}$$

The length of the curve from $x = 0$ to $x = 1$ is

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \\ &= \int_0^1 \sqrt{1 + 8x} \, dx \\ &= \int_0^1 (1 + 8x)^{\frac{1}{2}} \, dx \\ &= \left[\frac{(1 + 8x)^{\frac{1}{2} + 1}}{\left(\frac{1}{2} + 1\right) \cdot 8} \right]_0^1 \\ &= \frac{2}{3} \cdot \frac{1}{8} [(1 + 8x)^{\frac{3}{2}}]_0^1 \\ &= \frac{1}{12} [(1 + 8 \times 1)^{\frac{3}{2}} - (1 + 8 \times 0)^{\frac{3}{2}}] \\ &= \frac{1}{12} [(3^2)^{\frac{3}{2}} - 1] \\ &= \frac{1}{12} (27 - 1) \\ &= \frac{26}{12} \\ &= \frac{13}{6}. \end{aligned}$$

Dealing with Discontinuities in $\frac{dy}{dx}$

At a point on a curve where $\frac{dy}{dx}$ does not exist, $\frac{dx}{dy}$ may exist and we may be able to find the curve's length by expressing x as a function of y and applying the following formula.

Formula for the length of a smooth curve $x = g(y)$, $c \leq y \leq d$.

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$= \int_c^d \sqrt{1 + (g'(y))^2} dy.$$

Length of Parametric Curves

If the equation of the curve is given in the parametric form

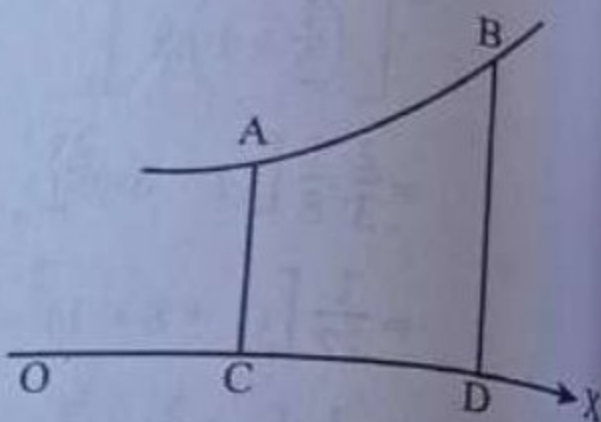
$x = \phi(t)$ and $y = \psi(t)$ then

$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ gives the length of the arc between the two points on it from $t = t_1$ to $t = t_2$.

Volume and Surface Integral

Solids of Revolution

Let AB be an arc. Draw AC and BD perpendicular to OX . When the region $ABCD$ revolves about OX , a solid is formed. This is called the solid revolution of arc AB about OX .



Formula 1: (The volume of a solid revolution about y-axis.) The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0$, $0 \leq a \leq x \leq b$ about y-axis is

$$V = \int_a^b 2\pi x f(x) dx.$$

Example 5. The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Here, $y = f(x) = \sqrt{x}$
The limits of integration are $a = 0$ and $b = 4$

$$\begin{aligned} \text{Volume (V)} &= \int_a^b 2\pi x f(x) dx \\ &= 2\pi \int_0^4 x \cdot \sqrt{x} dx \\ &= 2\pi \int_0^4 x^{\frac{3}{2}} dx \\ &= 2\pi \left[\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} \right]_0^4 \\ &= 2\pi \left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}} \right]_0^4 \\ &= \frac{4\pi}{5} [x^{\frac{5}{2}}]_0^4 \\ &= \frac{4\pi}{5} [4^{\frac{5}{2}} - 0^{\frac{5}{2}}] \\ &= \frac{4\pi}{5} [2^5] \\ &= \frac{128\pi}{5}. \end{aligned}$$

Formula 2: (The volume of the solid revolution about the x-axis): The volume of the solid generated by revolving the region between the y-axis and the graph of a continuous function $x = f(y) \geq 0$, $0 \leq c \leq y \leq d$ about x-axis is

$$V = \int_c^d 2\pi y f(y) dy.$$

Example 6. The region bounded by the curve $y = \sqrt{x}$, the y-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution

The limits of integration: y runs from $c = 0$ to $d = 2$

Definition (Revolution about y-axis)

If $x = g(y) \geq 0$ is smooth on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$= \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2} dy.$$

Illustrative Examples

Example 1. Find the area of the region between the curves $y^2 = 4ax$ and $x^2 = 4ay$.

Solution

Here, first curve is $y^2 = 4ax$... (i)

and the second curve is $x^2 = 4ay$... (ii)

Solving (i) and (ii), we get

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\Rightarrow \frac{x^4}{16a^2} = 4ax$$

$$\text{or, } x^4 - 64a^3x = 0$$

$$\text{or, } x(x^3 - 64a^3) = 0$$

$$\text{or, } x = 0, 4a$$

$\therefore$ The required area of the bounded region between the curves (i) and (ii) is given by,

$$A = \int_0^{4a} (y_1 - y_2) dx$$

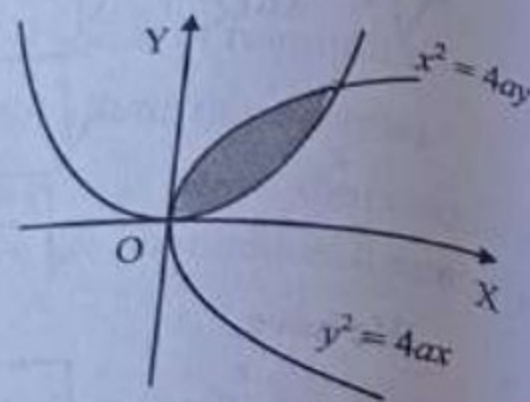
$$= \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a} \right) dx$$

$$= 2\sqrt{a} \int_0^{4a} x^{1/2} dx - \frac{1}{4a} \int_0^{4a} x^2 dx$$

$$= 2\sqrt{a} \cdot \frac{2}{3} [x^{3/2}]_0^{4a} - \frac{1}{4a} \cdot \frac{1}{3} [x^3]_0^{4a}$$

$$= \frac{16}{3} a^2 (2 - 1)$$

$$= \frac{16}{3} a^2 \text{ sq. units.}$$



Example 2.

Using integration, find the area of the circle $x^2 + y^2 = a^2$.

Solution

The equation of circle is

$$x^2 + y^2 = a^2 \quad \dots (i)$$

or, $y = \sqrt{a^2 - x^2}$

The centre of the given circle is at origin (0, 0) and its radius is a . Since the given circle is symmetrical about axes, x-axis and y-axis divide it into four equal parts as shown in the figure.

OAB is the portion of the circle lying in the first quadrant which is bounded by the given curve and the axes.

At A,

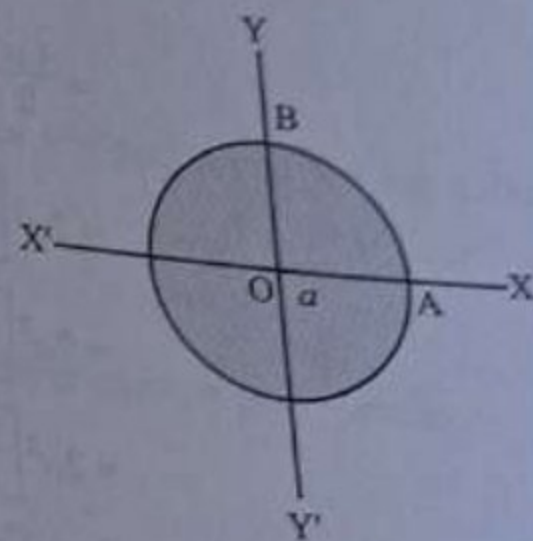
$$y = 0$$

$$x^2 = a^2$$

$$x = a$$

$$\text{At O, } x = y = 0$$

$$\text{Area of OAB} = \int_0^a y dx$$



Hence,

$$\text{Area of the circle} = 4 \times \text{Area of OAB}$$

$$= 4 \int_0^a y dx$$

$$= 4 \int_0^a \sqrt{a^2 - x^2} dx$$

$$\text{Put } x = a \sin \theta, \text{ then } dx = a \cos \theta d\theta$$

$$\text{When } x = 0, \sin \theta = 0$$

$$\text{or, } \sin \theta = 0$$

$$\therefore \theta = 0$$

$$\text{When } x = a,$$

$$a \sin \theta = a$$

$$\text{or, } \sin \theta = 1$$

$$\text{or, } \sin \theta = \sin \frac{\pi}{2}$$

$$\therefore \theta = \frac{\pi}{2}$$

$$\begin{aligned}
 \therefore \text{Area of the circle} &= 4 \int_0^{\pi/2} \sqrt{a^2 - a^2 \sin^2 \theta} \times a \cos \theta \, d\theta \\
 &= 4 \int_0^{\pi/2} a \cos \theta \times a \cos \theta \, d\theta \\
 &= 4a^2 \int_0^{\pi/2} \cos^2 \theta \, d\theta \\
 &= 4a^2 \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta \\
 &= \frac{4a^2}{2} \int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta \\
 &= 2a^2 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} \\
 &= 2a^2 \left[\left(\frac{\pi}{2} + \frac{\sin 2 \times \pi/2}{2} \right) - \left(0 + \frac{\sin 2 \times 0}{2} \right) \right] \\
 &= 2a^2 \left[\frac{\pi}{2} + \frac{\sin \pi}{2} - \frac{\sin 0}{2} \right] \\
 &= 2a^2 \left[\frac{\pi}{2} + \frac{0}{2} - \frac{0}{2} \right] \quad \left[\because \sin \pi = 0, \sin 0 = 0 \right] \\
 &= \pi a^2 \text{ sq. units.}
 \end{aligned}$$

Example 3. Find the area bounded by y-axis, the curve $x^2 = 4a(y - 2a)$ and $y = 6a$.

Solution

The given curve is $x^2 = 4a(y - 2a)$... (i)

At y-axis, $x = 0$.

Then, from (i),

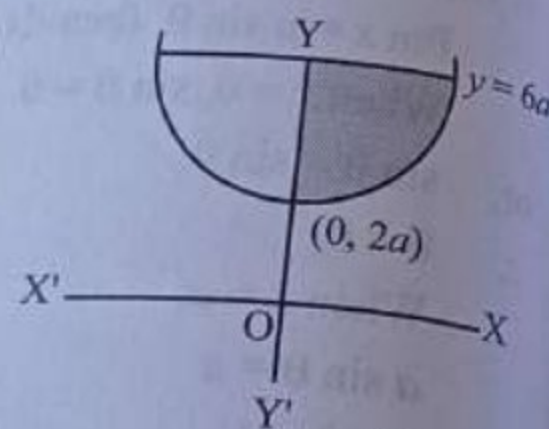
$$0 = 4a(y - 2a)$$

$$\Rightarrow y = 2a$$

Also, given line is $y = 6a$... (ii)

$\therefore$ Hence, the required area bounded by y-axis, the curve (i) and (ii) is $\int_{2a}^{6a} x \, dy$

$$\begin{aligned}
 &= \int_{2a}^{6a} \sqrt{4a(y - 2a)} \, dy \\
 &= 2\sqrt{a} \int_{2a}^{6a} (y - 2a)^{1/2} \, dy
 \end{aligned}$$



$$\begin{aligned}
 &= 2\sqrt{a} \left[\frac{(y - 2a)^{3/2}}{3/2 \cdot 1} \right]_{2a}^{6a} \\
 &= \frac{4\sqrt{a}}{3} [(y - 2a)^{3/2}]_{2a}^{6a} \\
 &= \frac{4\sqrt{a}}{3} [(6a - 2a)^{3/2} - (2a - 2a)^{3/2}] \\
 &= \frac{4\sqrt{a}}{3} (4a)^{3/2} \\
 &= \frac{4a^{1/2}}{3} \cdot 8a^{3/2} \\
 &= \frac{32a^2}{3} \text{ sq. units.}
 \end{aligned}$$

Example 4. Find the area between two curves $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{4}$.

Solution

Here the upper curve is $f(x) = \sec^2 x$ and lower curve is $g(x) = \sin x$.

The limits of integration are $a = 0$; $b = \frac{\pi}{4}$

$$\text{Area} = \int_0^{\pi/4} [f(x) - g(x)] \, dx$$

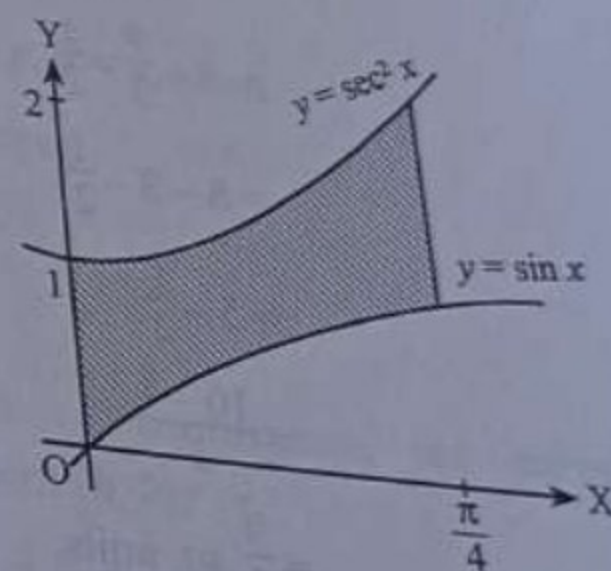
$$= \int_0^{\pi/4} (\sec^2 x - \sin x) \, dx$$

$$= [\tan x + \cos x]_0^{\pi/4}$$

$$= \left[\tan \frac{\pi}{4} + \cos \frac{\pi}{4} \right] - [\tan 0 + \cos 0]$$

$$= \left(1 + \frac{1}{\sqrt{2}} \right) - (0 + 1)$$

$$= \frac{1}{\sqrt{2}} \text{ sq. unit.}$$



Example 5. Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution

We first sketch the given curves. For limits of integration, we solve given curves, that is,

$$2 - x^2 = -x$$

$$\text{or, } x^2 - x - 2 = 0$$

$$(x+1)(x-2)=0$$

$$\therefore x = -1, 2.$$

The limits of integration are $a = -1$ and $b = 2$
Also the upper curve is $f(x) = 2 - x^2$
and lower curve is $g(x) = -x$

$$\text{Area (A)} = \int_{-1}^2 [f(x) - g(x)] dx$$

$$= \int_{-1}^2 [(2 - x^2) - (-x)] dx$$

$$= \int_{-1}^2 (2 - x^2 + x) dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^2$$

$$= \left[2 \times 2 + \frac{2^2}{2} - \frac{2^3}{3} \right] - \left[2 \times (-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3} \right]$$

$$= \left(4 + 2 - \frac{8}{3} \right) - \left(-2 + \frac{1}{2} - \frac{1}{3} \right)$$

$$= 6 - \frac{8}{3} + 2 - \frac{1}{2} + \frac{1}{3}$$

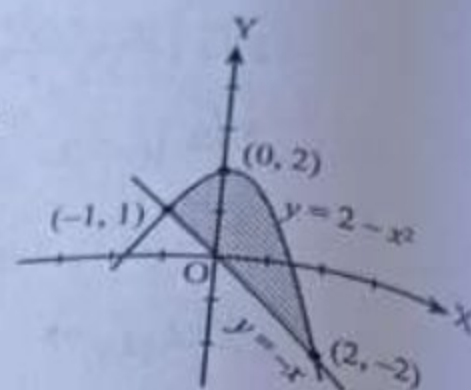
$$= 8 - \frac{9}{3} - \frac{1}{2}$$

$$= 8 - 3 - \frac{1}{2}$$

$$= 5 - \frac{1}{2}$$

$$= \frac{10-1}{2}$$

$$= \frac{9}{2} \text{ sq. units.}$$



Example 6. The line segment $x = 1 - y$, $0 \leq y \leq 1$, is revolved about the y -axis to generate the cone. Find its lateral surface area.

Solution

$$\text{Given, } c = 0, d = 1$$

$$x = 1 - y$$

$$\frac{dx}{dy} = -1.$$

Now,

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

We have,

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$= 2\pi \int_0^1 (1 - y) \sqrt{2} dy$$

$$= 2\sqrt{2}\pi \int_0^1 (1 - y) dy$$

$$= 2\sqrt{2}\pi \left[y - \frac{y^2}{2} \right]_0^1$$

$$= 2\sqrt{2}\pi \left[\left(1 - \frac{1^2}{2} \right) - \left(0 - \frac{0^2}{2} \right) \right]$$

$$= 2\sqrt{2}\pi \cdot \frac{1}{2}$$

$$= \pi\sqrt{2}.$$

Exercise 4.5

- Find the area enclosed by the curve $y = 3x$, the x -axis and ordinates at $x = 0$ and $x = 4$.
- Find the area bounded by the curve $y = \sin x$, $x = 0$, $x = \pi$.
- Find the area of the region between the curves
 - $y^2 = 16x$ and the line $y = 2x$
 - $y = x^2 - 2$ and $y = 2$.
- Find the area bounded by the axis of coordinates, the curve $x^2 = 4a(y - 2a)$ and the ordinate of the point (h, k) .
- Find the area bounded by the parabola $y^2 = 4x$ and the y -axis between the points $y = 0$ to $y = 2$.
- Find the area of the region between the curve $y = 4 - x^2$ and $0 \leq x \leq 3$ and the x -axis.
- Find the area of the region between the x -axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$.
- Using integration, find the area of
 - $x^2 + y^2 = 36$
 - $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

9. Find the length of the curve
- $y = x^{\frac{1}{2}}$ from $x = 0$ to $x = 4$
 - $y = \left(\frac{x}{2}\right)^{\frac{1}{2}}$ from $x = 0$ to $x = 2$.
10. a. Using integration, find the circumference of the circle $x^2 + y^2 = r^2$.
 b. Find the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $3y = 8x$.
11. a. The region between the curve $y = \sqrt{x}$, $0 \leq x \leq 4$ and the x -axis is revolved about the x -axis to generate a solid. Find its volume.
 b. Find the volume of the solid generated by revolving the region between the y -axis and the curve $x = \frac{2}{y}$, $1 \leq y \leq 4$ about the y -axis.
 c. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.
12. a. Find the lateral surface area of the cone generated by revolving the line segment $y = \frac{x}{2}$, $0 \leq x \leq 4$, about the x -axis.
 b. Find the lateral surface area of the cone generated by revolving the line segment $y = \frac{x}{2}$, $0 \leq x \leq 4$ about the y -axis.

Very Short Answer Questions

- Find the area bounded by the curve $y = 2x$, the x -axis and the ordinates at $x = 0$ and $x = 3$.
- Write the area bounded by the curve $x = f(y)$, the y -axis and the lines at $y = a$ and $y = b$.
- Find the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.
- Write the formula for the area between two curves $y = f(x)$ and $y = g(x)$ and two ordinates at $x = a$ and $x = b$.
- If f is smooth on $[a, b]$ then write the formula for the length of the curve $y = f(x)$ from a to b .
- Find the volume of solid obtained by rotating about the y -axis the region between $y = x$ and $y = x^2$.

Very Short Answer Questions

4.8 Trapezoidal and Simpson's Rule for Numerical Integration

Introduction

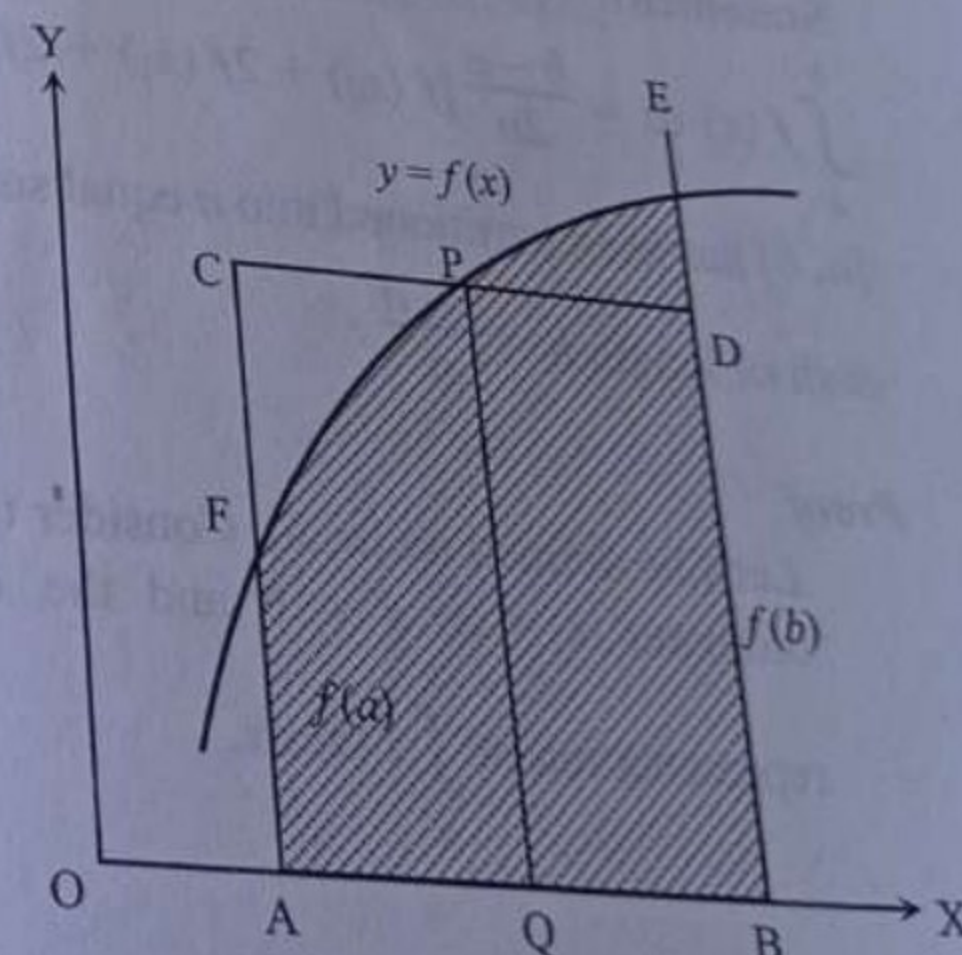
The antiderivatives of some functions like $\sin(x^2)$ and $\sqrt{1+x^3}$ have no elementary formulas to evaluate them. When we cannot find an antiderivative for a function f that we have to integrate, we can partition the interval of integration and replace f by a closely fitting polynomial on each subinterval, integrate the polynomials, and add the results to approximate the integral of f . This procedure is a numerical integration. In this section, we discuss two methods. They are *Trapezoidal Rule* and *Simpson's Rule*. We assume that f is positive and should be continuous on the interval of integration $[a, b]$.

Mid Point Rule (Rectangle Rule)

The integral $\int_a^b f(x) dx$ is represented by the area of the shaded region in the figure. The shaded area ABDF can be approximated by

$$A \approx (b-a)f\left(\frac{a+b}{2}\right)$$

which is the area of ABDC. In the figure, $OA = a$, $OB = b$,



$OQ = \frac{a+b}{2}$ where Q is the midpoint of the rectangle. Then, the area A can be approximated by the area of rectangle $ABDC$ and the area of $\frac{1}{2}f\left(\frac{a+b}{2}\right) \cdot (b-a)$.

$$\text{Thus, } \int_a^b f(x) dx = (b-a)f\left(\frac{a+b}{2}\right).$$

Trapezoidal Rule (Simple Form)

The value of the integral $\int_a^b f(x) dx$ is equal to the area bounded by the curve $y = f(x)$ between the ordinates $x = a$ and $x = b$. This can also be approximated by the area of trapezium as shown in the figure. The area of the trapezium is given by

$$\frac{1}{2}(b-a)[f(a)+f(b)]$$

$$\text{Thus, } \int_a^b f(x) dx = \frac{1}{2}(b-a)[f(a)+f(b)].$$

This is the simplest method of the trapezoidal rule to approximate the area bounded by the curve $y = f(x)$ between the ordinates $x = a$ to $x = b$.

Trapezoidal Rule (Composite Form)

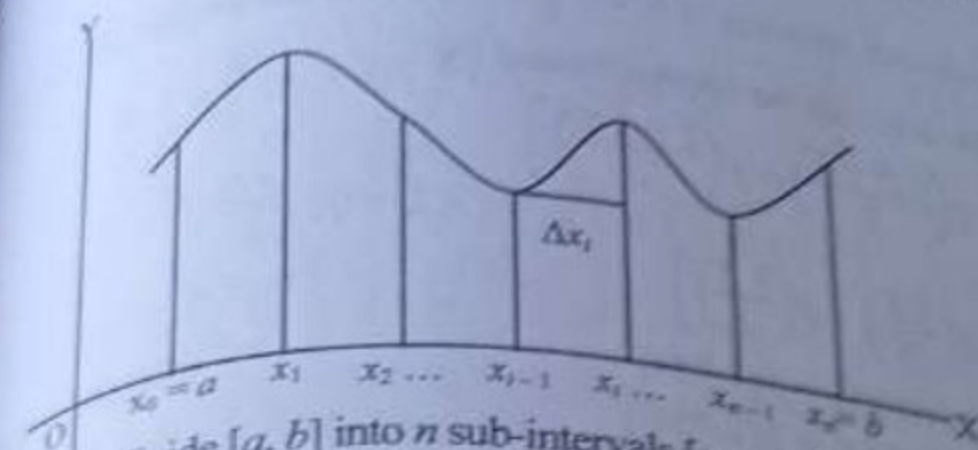
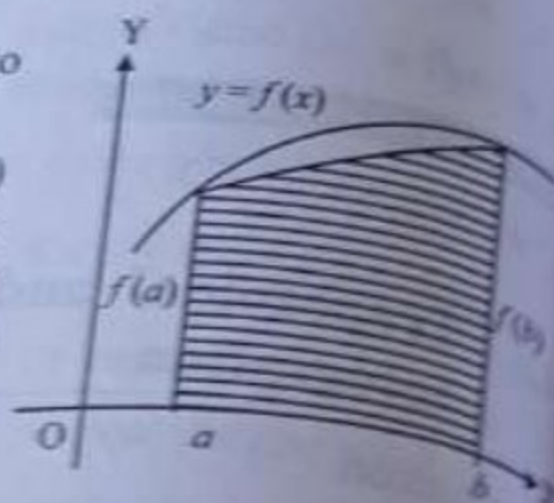
Statement: If a function f is continuous on the closed interval $[a, b]$, then

$$\int_a^b f(x) dx = \frac{b-a}{2n} [f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)] \text{ where}$$

$[a, b]$ has been partitioned into n equal subintervals $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$, each of length $h = \frac{b-a}{n}$.

Proof

Let f be defined on $[a, b]$. Consider the area bounded by the graph of f , the ordinates $x = a$, $x = b$ and the x -axis. This area is the geometrical representation of $\int_a^b f(x) dx$.



Let us divide $[a, b]$ into n sub-intervals $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$, each of length $h = \frac{b-a}{n}$. The ordinates corresponding to the points of subdivisions $x_0, x_1, x_2, \dots, x_{n-1}, x_n$ are $f(x_0), f(x_1), f(x_2), \dots, f(x_{n-1}), f(x_n)$. If we join the consecutive points on the graph by the line segments, we get n trapezoids. The sum of areas of these n trapezoids is an approximation to the area under the curve. We know that, the area of a trapezium = $\frac{1}{2}$ (sum of parallel sides) (distance between them).

So,

$$A = \frac{1}{2}[f(x_0)+f(x_1)]\Delta x + \frac{1}{2}[f(x_1)+f(x_2)]\Delta x + \dots + \frac{1}{2}[f(x_{n-1})+f(x_n)]\Delta x$$

$$= \frac{1}{2}[f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)] \frac{b-a}{n}$$

$$\text{Hence, } \int_a^b f(x) dx \approx \frac{b-a}{2n} [f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)].$$

Example 1. Use the (a) rectangle rule with one interval (b) rectangle rule with two sub intervals (c) fundamental theorem of calculus, to estimate $\int_1^2 3x^3 dx$. Compare the estimate with the exact value.

Solution

1. Here $f(x) = 3x^3$.

Now,

$$\int_1^2 3x^3 dx \approx (2-1)f\left(\frac{2+1}{2}\right)$$

$$= f\left(\frac{3}{2}\right)$$

$$= 3 \times \left(\frac{3}{2}\right)^3$$

$$= 10.125$$

- b. The length of each sub interval is $h = \frac{2-1}{2} = 0.5$. So the points are 1.0, 1.5, 2.0. Therefore the area can be approximated by

$$\begin{aligned}\int_1^2 3x^3 dx &= h \left[f\left(\frac{1.5}{2}\right) + f\left(\frac{2+1.5}{2}\right) \right] \\ &= h \left[f\left(\frac{1.5}{2}\right) + f\left(\frac{3.5}{2}\right) \right] \\ &= 0.5 [f(0.75) + f(1.75)] \\ &= 0.5 [3(1.25)^3 + 3(1.75)^3] \\ &= 0.5 [3(1.953) + 3(5.359)] \\ &= 0.5(21.920) \\ &= 10.96\end{aligned}$$

Thus, if we increase the number of intervals, the estimate becomes more accurate.

c. Exact value $= \int_1^2 3x^3 dx$

$$\begin{aligned}&= \left[3 \times \frac{x^4}{4} \right]_1^2 \\ &= 3 \left[\frac{2^4}{4} - \frac{1}{4} \right] \\ &= 3 \times 3.75 \\ &= 11.25.\end{aligned}$$

Example 2. Evaluate $\int_0^2 y dx$ by trapezoidal rule, with the values given in the table:

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	10	8	6	4	1

Solution

Here,

$$a = 0, b = 2, n = 4$$

We have,

$$h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2} = 0.5.$$

Also, given,

$$y_0 = 10, y_1 = 8, y_2 = 6, y_3 = 4, y_4 = 1.$$

By the trapezoidal rule

$$\begin{aligned}\int_0^2 y dx &= \frac{h}{2} (y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) \\ &= \frac{0.5}{2} (10 + 2 \times 8 + 2 \times 6 + 2 \times 4 + 1) \\ &= 11.75.\end{aligned}$$

Error Analysis

The error is the difference between the approximate value and the actual value of the definite integral $\int_a^b f(x) dx$. The following theorems give formulas for estimating the errors.

Theorem: If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_r in the trapezoidal approximation of the integral of f from a to b for n sub-intervals satisfies the inequality

$$E_r \leq \frac{M(b-a)^3}{12n^2}.$$

Example 3. Compute an approximate value of $\int_0^1 (1+x^2)^{-1} dx$ by using the composite trapezoid rule with three points. Then compare with the actual value of the integral. Next, determine the error formula and numerically verify an upper bound on it.

Solution

Here, $a = 0, b = 1$

For three points, $n = 2$

$$\text{Then, } h = \frac{b-a}{n} = \frac{1-0}{2} = 0.5$$

Now, 3 points to be considered are $x_0 = 0, x_1 = 0.5, x_2 = 1$. The value of the function at these points are

x	0	0.5	1
$y = f(x) = \frac{1}{1+x^2}$	1	0.8	0.5

Using Trapezoidal rule, we have,

$$\begin{aligned}\int_0^1 (1+x^2)^{-1} dx &\approx \frac{h}{2} (y_0 + 2y_1 + y_2) \\ &= \frac{0.5}{2} [1 + 2 \times 0.8 + 0.5] \\ &= 0.25 \times 3.1 = 0.775\end{aligned}$$

$$\begin{aligned}\text{Actual value} &= \int_0^1 \frac{1}{1+x^2} dx \\ &= [\tan^{-1} x]_0^1 \\ &= \tan^{-1}(1) - \tan^{-1}(0) \\ &= \frac{\pi}{4} - 0 \\ &= 0.785\end{aligned}$$

$$\text{Error in Trapezoidal rule} = 0.785 - 0.775 = 0.01$$

$$\text{Here, } f(x) = \frac{1}{1+x^2}$$

$$f'(x) = \frac{-2x}{(1+x^2)^2}$$

$$f''(x) = \frac{2(3x^2-1)}{(1+x^2)^3}$$

The maximum value of $|f''(x)|$ occurs at $x = 1$. It is 2.

$$\therefore M = 2$$

$$\text{Error bounds} = \frac{(b-a)^3 \cdot M}{12n^2}$$

$$= \frac{(1-0)^3 \cdot 2}{12 \times 2^2}$$

$$= \frac{1}{24}$$

$$= 0.0417 \text{ (upper bound).}$$

Hence the error is within the bound.

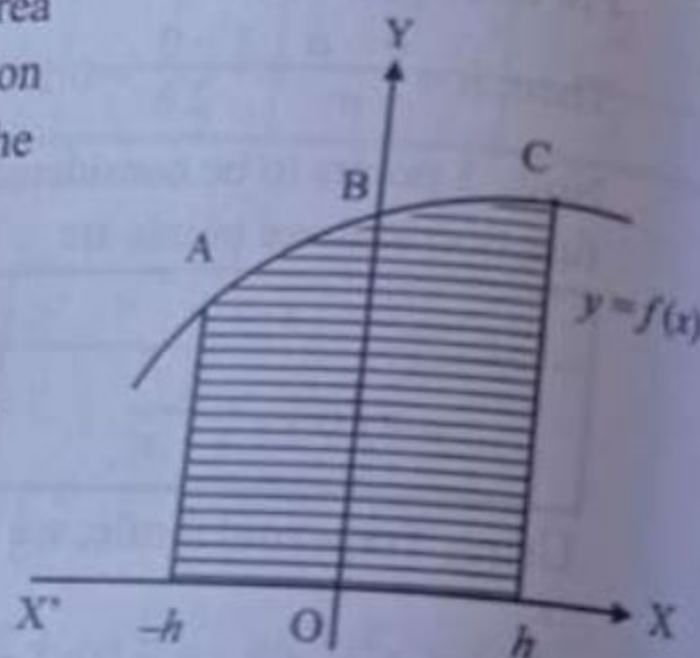
4.9 Simpson's Rule

For the Simpson's rule, we find the area under a parabolic arc. Here, approximation is obtained by joining the ordinates with the arcs of parabolas.

Let the parabola be $y = ax^2 + bx + c$.

Draw the ordinates to this curve at $x = -h, 0, h$ and denote them by y_0, y_1, y_2 respectively.

The area under the parabola, above the x -axis and bounded by the ordinates y_0 and y_2 is given by



$$\begin{aligned}\text{Area (A)} &= \int_{-h}^h y dx \\ &= \int_{-h}^h (ax^2 + bx + c) dx \\ &= \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx \right]_{-h}^h \\ &= \frac{h}{3} (2ah^2 + 6c).\end{aligned}$$

Since the three points $A(-h, y_0)$, $B(0, y_1)$, $C(h, y_2)$ lie on the parabola, we have,

$$y_0 = ah^2 + bh + c,$$

$$y_1 = c,$$

$$y_2 = ah^2 - bh + c$$

Then,

$$y_0 + y_2 = 2ah^2 + 2c$$

$$\text{and } y_0 + 4y_1 + y_2 = 2ah^2 + 2c$$

$$\therefore A = \frac{h}{3} (y_0 + 4y_1 + y_2).$$

This is the simplest Simpson's rule.

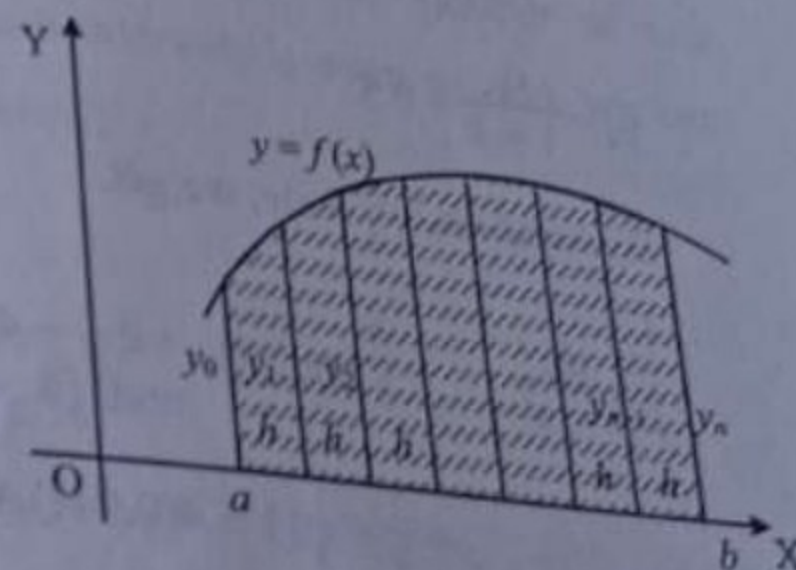
Now, let us divide the area into an even number n of vertical strips each of width $h = \frac{b-a}{n}$. Applying the above formula to each successive double strip, we get

$$A = \frac{h}{3} (y_0 + 4y_1 + y_2) + \frac{h}{3} (y_2 + 4y_3 + y_4) + \dots + \frac{h}{3} (y_{n-2} + 4y_{n-1} + y_n).$$

On simplification, we get,

$$\int_a^b y dx = \frac{h}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \dots + 4y_{n-1} + y_n).$$

This is Simpson's $\frac{1}{3}$ rule, due to Thomas Simpson (1710 – 1761). Here, the number of subintervals is even.



Example 4. Evaluate $\int_0^1 \frac{1}{1+x^2} dx$ by Simpson's $\frac{1}{3}$ Rule, taking $n = 4$. Also find the error.

Solution

Here, $a = 0$, $b = 1$, $n = 4$

We have $h = \frac{b-a}{n} = \frac{1}{4}$

Hence, $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$, $x_4 = 1$.

Since, $y = \frac{1}{1+x^2}$, we have,

$$y_0 = \frac{1}{1+0} = 1$$

$$y_1 = \frac{1}{1+(\frac{1}{4})^2} = 0.941$$

$$y_2 = \frac{1}{1+(\frac{1}{2})^2} = 0.8$$

$$y_3 = \frac{1}{1+(\frac{3}{4})^2} = 0.64$$

$$y_4 = \frac{1}{1+1} = 0.5$$

By Simpson's $\frac{1}{3}$ Rule, we get,

$$\begin{aligned} \int_0^1 \frac{dx}{1+x^2} &= \frac{h}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) \\ &= \frac{1}{4} \times \frac{1}{3} [1 + 4(0.941) + 2(0.8) + 4(0.64) + 0.5] \\ &= 0.7853. \end{aligned}$$

$$\text{Exact value} = \int_0^1 \frac{1}{1+x^2} dx$$

$$= [\tan^{-1} x]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4}$$

Thus, the exact value of the integral is $\frac{\pi}{4}$, which is 0.7854.

The error is $0.7854 - 0.7853 = 0.0001$.

Example 5. Evaluate $\int_0^2 \sqrt{1+x^2} dx$ by Simpson's $\frac{1}{3}$ rule, taking $n = 4$.

Solution

Here, $a = 0$, $b = 2$, $n = 4$.

We have, $h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$

Here, $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$

$y_0 = 1$, $y_1 = \sqrt{1+(0.5)^2} = 1.060$,

$y_2 = 1.414$,

$y_3 = 2.091$,

$y_4 = 3$

By Simpson's $\frac{1}{3}$ rule, we have,

$$\begin{aligned} \int_0^2 \sqrt{1+x^2} dx &= \frac{h}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) \\ &= \frac{0.5}{3} (1 + 4.240 + 2.828 + 8.364 + 3) \\ &= 3.240. \end{aligned}$$

The composite Simpson's $\frac{1}{3}$ rule can be used when the number of sub-intervals is even. So, when number of sub-intervals is even or odd, we can use Simpson's $\frac{3}{8}$ rule. The rule is given below.

Simpson's $\frac{3}{8}$ Rule

If a function $y = f(x)$ is continuous on $[a, b]$ then

$$\int_a^b y dx = \frac{3h}{8} (y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \dots + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)$$

where $[a, b]$ is divided into n equal subintervals $[x_0, x_1]$, $[x_1, x_2]$... $[x_{n-1}, x_n]$ each of the length $\frac{b-a}{n}$.

Example 6. Use Simpson's $\frac{3}{8}$ rule to evaluate $\int_0^{\pi/2} \sqrt{\sin x} dx$ with 3 subintervals.

Solution

In this case, we have four points and 3 subintervals.

$$\therefore h = \frac{\frac{\pi}{2} - 0}{3} = \frac{\pi}{6} \text{ and } y = \sqrt{\sin x} = f(x).$$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y	0	0.70711	0.93060	1

Using Simpson's $\frac{3}{8}$ rule

$$\begin{aligned}\int_0^{\pi/2} \sqrt{\sin x} \, dx &= \frac{3h}{8} [y_0 + 3y_1 + 3y_2 + y_3] \\ &= \frac{3 \times \frac{\pi}{6}}{8} [0 + 3 \times 0.70711 + 3 \times 0.93060 + 1] \\ &= \frac{\pi}{16} \times 5.91313 \\ &= 1.16104.\end{aligned}$$

Error Analysis

Theorem : If $f^{(4)}$ is continuous and M is any upper bound (maximum value) for the values of $|f^{(4)}|$ on $[a, b]$, then the error E_S

$$E_S \leq \frac{M(b-a)^5}{180n^4}.$$

Illustrative Examples

Example 1. Evaluate $\int_0^{\pi} \sin x \, dx$ with $n = 4$ using trapezoidal rule.

Solution

Here, $y = \sin x$, $n = 4$, $h = \frac{\pi}{4}$, $a = 0$, $b = \pi$.

The points on the x -axis and y -axis are shown in the table.

End point x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$y = \sin x$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0

By the trapezoidal rule, we have,

$$\begin{aligned}\int_0^{\pi} \sin x \, dx &\approx \frac{h}{2} (y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) \\ &= \frac{\pi}{8} \left(0 + 2\left(\frac{1}{\sqrt{2}}\right) + 2(1) + 2\left(\frac{1}{\sqrt{2}}\right) + 0 \right) \\ &= \frac{\pi}{8} (2\sqrt{2} + 2) \\ &= 1.896.\end{aligned}$$

Example 2. Using the trapezoidal rule, compute $\int_0^2 (2x^2 - 1) \, dx$ with 4 intervals. Find the absolute error of approximation from its actual value.

Solution

Here, $a = 0$, $b = 2$, $n = 4$

$$h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$$

The five points to be considered are $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$. The values of function at these points are tabulated below

End point	$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1$	$x_3 = 1.5$	$x_4 = 2$
$y = 2x^2 - 1$	-1	-0.5	1	3.5	7

Hence, by using trapezoidal rule, we have

$$\begin{aligned}\int_0^2 (2x^2 - 1) \, dx &= \frac{h}{2} [y_0 + 2y_1 + 2y_2 + 2y_3 + y_4] \\ &= \frac{0.5}{2} [-1 + 2 \times (-0.5) + 2 \times 1 + 2 \times 3.5 + 7] \\ &= 0.25 [-1 - 1 + 2 + 7 + 7] = 3.5\end{aligned}$$

$$\text{Actual value} = \int_0^2 (2x^2 - 1) \, dx$$

$$\begin{aligned}&= \left[\frac{2x^3}{3} - x \right]_0^2 \\ &= \frac{2 \times 2^3}{3} - 2\end{aligned}$$

$$= 3.3333$$

$$\begin{aligned}\text{Absolute error} &= |3.3333 - 3.5| \\ &= 0.1667.\end{aligned}$$

Example 3. Using Simpson's $\frac{1}{3}$ rule, calculate $\int_1^5 x^4 \, dx$ with $n = 4$.

Solution

Here, $y = f(x) = x^4$

$a = 1$, $b = 5$, $n = 4$

$$\text{We have, } h = \frac{b-a}{n} = \frac{5-1}{4} = 1$$

The five points to be considered are $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 5$. The values of the function at these points are tabulated below.

End point	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$	$x_3 = 4$	$x_4 = 5$
$y = f(x) = x^4$	1	16	81	256	625

Using Simpson's $\frac{1}{3}$ rule, we have

$$\begin{aligned}\int_1^5 x^4 dx &= \frac{h}{3} [y_0 + 4y_1 + 2y_2 + 4y_3 + y_4] \\ &= \frac{1}{3} [1 + 4 \times 16 + 2 \times 81 + 4 \times 256 + 625] \\ &= 625.33.\end{aligned}$$

Example 4. Evaluate the following integral using Simpson's rule: $\int_0^\pi \sin x \, dx$, $n=6$.

Solution

$$\text{Here, } y = f(x) = \sin x$$

$$a = 0, b = \pi, n = 6$$

We have,

$$h = \frac{b-a}{n} = \frac{\pi-0}{6} = \frac{\pi}{6}$$

The seven points to be considered are $x_0 = 0, x_1 = \frac{\pi}{6}$,

$x_2 = \frac{\pi}{3}, x_3 = \frac{\pi}{2}, x_4 = \frac{2\pi}{3}, x_5 = \frac{5\pi}{6}$ and $x_6 = \pi$. The values of the function at these points are tabulated below:

End point	$x_0 = 0$	$x_1 = \frac{\pi}{6}$	$x_2 = \frac{\pi}{3}$	$x_3 = \frac{\pi}{2}$	$x_4 = \frac{2\pi}{3}$	$x_5 = \frac{5\pi}{6}$	$x_6 = \pi$
$y = f(x) = \sin x$	0	0.5	0.8660	1	0.8660	0.5	0

Using Simpson's $\frac{1}{3}$ rule, we have

$$\begin{aligned}\int_0^\pi \sin x \, dx &= \frac{h}{3} [y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6] \\ &= \frac{\pi/6}{3} [0 + 4 \times 0.5 + 2 \times 0.8660 + 4 \times 1 + 2 \times 0.8660 + 4 \times 0.5 + 0] \\ &= \frac{\pi}{18} (11.464) \\ &= 2.0008.\end{aligned}$$

Example 5. Using Simpson's $\frac{1}{3}$ rule, evaluate: $\int_0^1 \sqrt{1+2x^2} \, dx$, $h = 0.25$.

Solution

$$\text{Here, } f(x) = \sqrt{1+2x^2}$$

$$a = 0, b = 1, h = 0.25$$

We have,

$$h = \frac{b-a}{n}$$

$$0.25 = \frac{1-0}{n}$$

$$n = 4$$

The five points to be considered are $x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$. The values of the function at these points are tabulated below:

End point	$x_0 = 0$	$x_1 = 0.25$	$x_2 = 0.5$	$x_3 = 0.75$	$x_4 = 1$
$y = f(x) = \sqrt{1+2x^2}$	1	1.0607	1.2247	1.4577	1.7321

Using Simpson's $\frac{1}{3}$ rule, we have

$$\begin{aligned}\int_0^1 \sqrt{1+2x^2} \, dx &= \frac{h}{3} [y_0 + 4y_1 + 2y_2 + 4y_3 + y_4] \\ &= \frac{0.25}{3} [1 + 4 \times 1.0607 + 2 \times 1.2247 + 4 \times 1.4577 + 1.7321] \\ &= 1.2712.\end{aligned}$$

Exercise 4.6

1. Use composite trapezoidal rule to evaluate

a. $\int_0^3 2x \, dx$, $n = 3$

b. $\int_1^2 x^{-2} \, dx$, $h = \frac{1}{2}$

c. $\int_0^1 \frac{dx}{1+x^2}$, $n = 2$

d. $\int_0^\pi \sin x \, dx$, $n = 4$

2. Use the trapezoidal rule with $n = 4$ to estimate $\int_1^2 x^2 \, dx$. Compare the estimate with the exact value of the integral.

3. Using composite trapezoidal rule, evaluate $\int_0^1 \frac{1}{1+x^2} \, dx$ with three points of the intervals. Find the error of approximation. How many points are to be considered to make the approximated value within 0.0001?

4. Evaluate using Simpson's $\frac{1}{3}$ rule.

a. $\int_0^1 5x^4 dx, n=4$

b. $\int_0^{0.2} \sqrt{1-2x^2} dx$

c. $\int_0^{\pi} \sin x dx, n=6$

5. Using Simpson's $\frac{1}{3}$ rule, evaluate $\int_0^1 \frac{1}{1+x} dx$ with three points of the intervals. Find the error of approximation. How many points are to be considered to make the approximated value within 10^{-4} ?6. Using Simpson's $\frac{3}{8}$ rule, evaluate $\int_1^4 e^x \ln x dx, n=3$.**Very Short Answer Questions**

- To approximate $\int_a^b f(x) dx$, the interval $[a, b]$ has divided into equal n subintervals then write the length of each interval (h).
- Find the approximate value of $\int_0^4 x^2 dx$ using mid-point rule with one interval.
- Write Trapezoidal rule.
- Write Simpson's $\frac{1}{3}$ rule.

Answers

- a. 9 b. 0.535 c. 0.775 d. 1.896
- Approximate value = 2.34375; Exact value = 2.33333
- 0.775, 0.0104, 42 or more points
- a. 1.0026 b. 0.1973 c. 2.0008
- 0.6944, Error = 0.0013, 8 or more points 6. 58.9698

Very Short Answer Questions

2. 16

Differential Equations

UNIT
5

5.1 Introduction

There are various physical problems in our surroundings. We also call them engineering problems. To solve such type of problems, we have to formulate the problems in terms of mathematical expressions containing variables, functions, equations etc. This type of expression is called the mathematical model of the problem. The process of setting up a model, solving it and interpreting the result is called mathematical modeling.

Many physical problems are based on velocity and acceleration (i.e. on derivatives). Thus, a model is an equation containing derivatives of an equation containing derivatives of an unknown function. Such an equation is called a differential equation.

5.2 Differential Equation

An equation involving differentials or differential coefficients (derivatives) of a function is called a differential equation.

If the function, whose derivatives are used in the equation, is of a single independent variable then it is called an **ordinary differential equation (ODE)**.

For examples

i. $\frac{dy}{dx} + y = x$

ii. $y'' + 6y' + 5y = 6x$ etc.

When an equation contains the derivative of a function with respect to two or more than two independent variables then it is called a **partial differential equation (PDE)**.

For examples

i. $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = \lambda f$

ii. $\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}$ etc.

5.3 Order and Degree of Differential Equation

Order of ODE

The order of the highest derivative of the function present in the differential equation is called the order of the ODE.

For examples

i. $\frac{dy}{dx} = x^5$ is an ODE of order 1.

ii. $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 4 = 0$ is an ODE of order 2.

Degree of ODE

The degree of the highest derivative of the function present in the differential equation after making free from fractions and the radicals is called the degree of an ODE.

For examples

i. $\frac{dy}{dx} + \frac{y}{x} = e^x$ is an ODE of degree 1.

ii. $(y'')^5 + 2y' = e^{-x}$ is an ODE of degree 5.

iii. $\left(\frac{dy}{dx}\right)^2 - 5\left(\frac{d^2y}{dx^2}\right)^3 + 4 = 0$ is an ODE of degree 3.

Formation of Differential Equation

A differential equation can be formed by eliminating the arbitrary constants, which can be done by successive differentiation.

Example 1. Form a differential equation from $y = a \cos x + b \sin x$.**Solution**Differentiating both sides w.r. to x , we get,

$$\frac{dy}{dx} = -a \sin x + b \cos x$$

Again, differentiating both sides w.r. to x , we get,

$$\begin{aligned} \frac{d^2y}{dx^2} &= -a \cos x - b \sin x \\ &= -(a \cos x + b \sin x) \\ &= -y \end{aligned}$$

$\therefore \frac{d^2y}{dx^2} + y = 0$ which is the required differential equation.

Solution of a Differential Equation

Any relation between the variables which is free from the derivatives and satisfies the given differential equation identically is called the solution of the differential equation.

For example: $y = a \cos x + b \sin x$ is solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

General and Particular Solution

If a solution consists the number of arbitrary constants equal to the order of the differential equation then the solution is said to be **general solution**.

The solution obtained from the general solution by giving particular values to one or more of the arbitrary constants is called a **particular solution**.

For example: $y = cx$ is the general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x}$.

If we set the condition $y(1) = 1$ then the solution becomes $y = x$ which is the particular solution of $\frac{dy}{dx} = \frac{y}{x}$.

A particular solution is obtained from a general solution by an initial condition $y(x_0) = y_0$, with given values x_0 and y_0 . It is used to determine a value of arbitrary constant. An ODE together with an initial condition is called **initial value problem**.

5.4 Differential Equation of First Order and First Degree

A differential equation of the form $M(x, y) dx + N(x, y) dy = 0$ is called a differential equation of first order and first degree. The general solution of this type of equation contains only one arbitrary constants.

NOTE We can not solve all differential equations.

In this chapter, we will discuss the following forms of ordinary differential equations of first order and first degree.

1. Variable separation form
2. Homogeneous form
3. Exact form
4. Linear form

5.5 Differential Equations with Separable Variables

If the equation

$$M(x, y) dx + N(x, y) dy = 0 \quad \dots(i)$$

can be written in the form

$$f(x) dx + f(y) dy = 0 \quad \dots(ii)$$

then the equation is in the **variable separation form**. To obtain the solution, we integrate both sides of (ii)

$$\int f(x) dx + \int f(y) dy = c \quad \dots(iii)$$

where c is an arbitrary constant of integration.

If we give particular value to the constant c then the solution is called particular solution.

Illustrative Examples

Example 1. Solve: $\frac{dy}{dx} = -\frac{y}{x}$.

Solution

Given,

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\text{or, } \frac{dy}{y} = -\frac{dx}{x}$$

$$\text{or, } \frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating both sides, we have,

$$\log x + \log y = A$$

$$\text{or, } \log(xy) = \log c$$

$$\therefore xy = c.$$

Example 2. Solve: $\frac{dy}{dx} = \frac{x}{y}$.

Solution

The equation can be written as,

$$y dy = x dx$$

$x dx - y dy = 0$
Integrating both sides, we have,

$$\frac{x^2}{2} - \frac{y^2}{2} = \frac{c}{2}$$

$$x^2 - y^2 = c.$$

Example 3. Solve: $y' = 1 + y^2$.

Solution

$$\frac{dy}{dx} = 1 + y^2$$

$$\text{or, } \frac{dy}{1 + y^2} = dx$$

Integrating both sides, we have,

$$\int \frac{dy}{1 + y^2} = \int dx + c$$

$$\text{or, } \tan^{-1} y = x + c$$

$$\text{or, } y = \tan(x + c)$$

$$\therefore y = \tan(x + c).$$

Example 4. Solve: $\frac{dy}{dx} = \frac{xy + y}{xy + x}$.

Solution

$$\frac{dy}{dx} = \frac{xy + y}{xy + x}$$

$$\text{or, } \frac{dy}{dx} = \frac{y(x + 1)}{x(y + 1)}$$

$$\text{or, } x(y + 1) dy = y(x + 1) dx$$

$$\text{or, } \left(\frac{y+1}{y}\right) dy = \left(\frac{x+1}{x}\right) dx$$

$$\text{or, } \left(1 + \frac{1}{y}\right) dy = \left(1 + \frac{1}{x}\right) dx$$

Integrating both sides, we have,

$$y + \log y = x + \log x + c$$

$$\text{or, } \log y - \log x + y - x = c$$

$$\therefore \log \frac{y}{x} + y - x = c.$$

Example 5. Solve: $\sqrt{1-x^2} dy + \sqrt{1-y^2} dx = 0$.

Solution

Given,

$$\sqrt{1-x^2} dy + \sqrt{1-y^2} dx = 0$$

$$\text{or, } \frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0$$

Integrating, we have,
 $\sin^{-1} x + \sin^{-1} y = \sin^{-1} c$
 or, $\sin^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2}) = \sin^{-1} c$
 $\therefore x\sqrt{1-y^2} + y\sqrt{1-x^2} = c.$

Example 6. Solve: $\frac{dy}{dx} = e^{x-y} + x^3 \cdot e^{-y}.$

Solution

Here,

$$\frac{dy}{dx} = e^{x-y} + x^3 \cdot e^{-y}$$

$$\text{or, } \frac{dy}{dx} = e^x \cdot e^{-y} + x^3 e^{-y}$$

$$\text{or, } \frac{dy}{dx} = e^{-y} (e^x + x^3)$$

$$\text{or, } \frac{dy}{e^{-y}} = (e^x + x^3) dx$$

$$\text{or, } e^y dy = (e^x + x^3) dx$$

Integrating, we get,

$$\int e^y dy = \int (e^x + x^3) dx + c$$

$$\therefore e^y = e^x + \frac{x^4}{4} + c.$$

Example 7. Solve: $\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0.$

Solution

Integrating both sides, we get,

$$\int \frac{dx}{1+x^2} + \int \frac{dy}{1+y^2} = \tan^{-1} c$$

$$\text{or, } \tan^{-1} x + \tan^{-1} y = \tan^{-1} c$$

$$\text{or, } \tan^{-1} \left(\frac{x+y}{1-xy} \right) = \tan^{-1} c$$

$$\text{or, } \frac{x+y}{1-xy} = c$$

$$\therefore x+y = c(1-xy).$$

Example 8. Solve: $x dy - y dx = 0, y(0) = 1.$

Solution

$$x dy - y dx = 0$$

$$\text{or, } x dy = y dx$$

$$\text{or, } \frac{dy}{y} = \frac{dx}{x}$$

Integrating both sides, we have,
 $\log x = \log y + \log c$
 $\log x = \log cy$... (i)

$$x = cy$$

$$\text{By given, } y(0) = 1$$

$$\text{when } x = 0, y = 1$$

Then, from (i), we have,

$$0 = c \cdot 1$$

$$c = 0$$

Substituting the value of c in (i), we get, $x = y$, which is the required solution.

Exercise 5.1

Find the order and degree of the following differential equations.

1. a. $\frac{dy}{dx} = 2$

b. $\frac{d^2y}{dx^2} = \sin x$

c. $x \frac{d^3y}{dx^3} + y + \left(\frac{dy}{dx}\right)^4 = 0$

d. $(y'')^3 + 4y' = e^x$

e. $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

2. Solve the following differential equations using separation of variables.

a. $x dx - y dy = 0$

b. $\frac{dy}{dx} = \frac{y}{x}$

c. $\frac{dy}{dx} = \frac{x^3 + 1}{y^3 + 1}$

d. $(1+x^2)y' = 1$

e. $y dx - x dy = xy dx$

f. $(xy^2 + x)dx + (yx^2 + y)dy = 0$

g. $\frac{dy}{dx} = \frac{e^x + 1}{y}$

h. $\frac{dy}{dx} = e^{x-y} + e^{-y}$

i. $e^{x-y} dx + e^{y-x} dy = 0$

j. $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

k. $\frac{dy}{dx} + \frac{1 + \cos 2y}{1 - \cos 2x} = 0$

3. Solve the initial value problems

a. $\frac{dy}{dx} = 2x - 7, y(2) = 0$

b. $\frac{dy}{dx} = 10 - x, y(0) = -1$

c. $\frac{dy}{dx} = 9x^2 - 4x + 5, y(-1) = 0$

Very Short Answer Questions

- Write the order of the differential equation $x \frac{d^4 y}{dx^4} + y + \left(\frac{dy}{dx}\right)^3 = 0$.
- Write the degree of the differential equation $(y''')^2 + 3y' = e^x$.
- Define ordinary differential equation.
- Find the solution of $\frac{dy}{dx} = x^2$.

Answers

- order = 1, degree = 1
 - order = 3, degree = 1
 - order = 2, degree = 2
- $x^2 - y^2 = c$
 - $x = cy$
 - $\frac{x^4}{4} + x = \frac{y^4}{4} + y + c$
 - order = 2, degree = 1
 - order = 2, degree = 3
- $y = \tan^{-1} x + c$
 - $\log\left(\frac{x}{y}\right) = x + c$
 - $(x^2 + 1)(y^2 + 1) = c$
 - $y^2 = 2e^x + 2x + c$
 - $e^y = e^x + x + c$
 - $e^{2x} + e^{2y} = c$
 - $\tan x \tan y = c$
 - $\cot x = \tan y + c$
- $y = x^2 - 7x + 10$
 - $y = 10x - \frac{x^2}{2} - 1$
 - $y = 3x^3 - 2x^2 + 5x + 10$

Very Short Answer Questions

- 4
- 2
- $y = \frac{x^3}{3} + c$

5.6 Homogeneous Differential Equations

A differential equation of the form $\frac{dy}{dx} = \frac{\phi(x, y)}{\Psi(x, y)}$, where $\phi(x, y)$ and $\Psi(x, y)$ are the homogeneous functions of x and y of same degree, is called a homogeneous differential equation.

In other words, a differential equation of the form $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$ is called a homogeneous differential equation.

For example, the differential equation $\frac{dy}{dx} = \frac{y}{x} + \tan \frac{y}{x}$ is homogeneous differential equation because it is in the form $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$.

But the equation $\frac{dy}{dx} = \frac{y+1}{x}$ is not homogeneous differential equation because it can not be written in the form $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$ as R.H.S is of the form $\frac{y}{x} + \frac{1}{x}$.

A very easy way to check whether a first degree and first order differential equation is homogeneous or not, put $y = vx$ and see whether or not all x 's cancel out or not. If all x 's cancel out then it is homogeneous, otherwise it is not homogeneous.

Procedures for Solving Homogeneous Equation

- Put $y = vx$ in the equation. Then, $\frac{dy}{dx} = v + x \frac{dv}{dx}$
- Substitute these values in the given equation $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$, we get

$v + x \frac{dv}{dx} = \phi(v)$
 $\Rightarrow \frac{dv}{\phi(v) - v} = \frac{dx}{x}$ which is variable separated form and we solve it by integrating both sides as in previous section.

Illustrative Examples

Example 1. Solve: $\frac{dy}{dx} = \frac{2x+y}{x}$.

Solution

The equation can be written as $\frac{dy}{dx} = 2 + \frac{y}{x}$ which is of the form $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$.

So, it is a homogeneous differential equation.

Put, $y = vx$.

Differentiating both sides, we get,

$$\begin{aligned} \frac{dy}{dx} &= v \frac{dx}{dx} + x \frac{dv}{dx} \\ &= v + x \frac{dv}{dx} \end{aligned}$$

Putting these values in given equation, we get,

$$v + x \frac{dv}{dx} = \frac{2x + vx}{x}$$

$$\text{or, } v + x \frac{dv}{dx} = 2 + v$$

$$\text{or, } x \frac{dv}{dx} = 2$$

$$\text{or, } dv = 2 \frac{dx}{x} \text{ (variables separated form)}$$

Integrating both sides, we have,

$$v = 2 \log x + \log c$$

$$\text{or, } v = \log x^2 + \log c$$

$$\text{or, } v = \log cx^2$$

$$\text{or, } \frac{y}{x} = \log cx^2$$

$$\therefore y = x \log cx^2.$$

Example 2. Solve: $xy \frac{dy}{dx} = x^2 + y^2$.

Solution

Given differential equation is

$$xy \frac{dy}{dx} = x^2 + y^2$$

$$\text{or, } \frac{dy}{dx} = \frac{x^2 + y^2}{xy} \quad \dots (i)$$

This is homogeneous differential equation.

Put $y = vx$, then

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

Then, from (i)

$$v + x \frac{dv}{dx} = \frac{x^2 + v^2 x^2}{x \cdot vx}$$

$$\text{or, } v + x \frac{dv}{dx} = \frac{1 + v^2}{v}$$

$$\text{or, } x \frac{dv}{dx} = \frac{1 + v^2}{v} - v$$

$$\text{or, } x \frac{dv}{dx} = \frac{1 + v^2 - v^2}{v}$$

$$\text{or, } x \frac{dv}{dx} = \frac{1}{v}$$

$$\text{or, } v dv = \frac{dx}{x}$$

Integrating, we have,

$$\frac{v^2}{2} = \log x + c$$

$$\text{or, } \left(\frac{y}{x}\right)^2 = 2(\log x + c)$$

$$\therefore y^2 = 2x^2(\log x + c).$$

Example 3. Solve: $\frac{dy}{dx} = \frac{y}{x} + \tan \frac{y}{x}$.

Solution

Here,

$$\frac{dy}{dx} = \frac{y}{x} + \tan \frac{y}{x} \quad \dots (i)$$

The given equation is of the form $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$.

So, it is homogeneous differential equation.

Put $y = vx$, then

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

Now, equation (i) becomes,

$$v + x \frac{dv}{dx} = \frac{vx}{x} + \tan \frac{vx}{x}$$

$$\text{or, } x \frac{dv}{dx} = \tan v$$

$$\text{or, } \frac{dv}{\tan v} = \frac{dx}{x}$$

$$\text{or, } \frac{\cos v}{\sin v} dv = \frac{dx}{x}$$

Integrating, we have,

$$\log \sin v = \log x + \log c$$

$$\text{or, } \log \sin v = \log cx$$

$$\text{or, } \sin v = cx$$

$$\therefore \sin \left(\frac{y}{x}\right) = cx.$$

Example 4. Solve $x^2 y dx = (x^3 + y^3) dy$.

Solution

Here,

$$x^2 y dx = (x^3 + y^3) dy$$

$$\text{or, } \frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$$

This is homogeneous differential equation.

Put $y = vx$ then

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

Hence equation (i) becomes,

$$v + x \frac{dv}{dx} = \frac{x^2 \cdot vx}{x^3 + v^3 x^3}$$

$$\text{or, } v + x \frac{dv}{dx} = \frac{v}{1 + v^3}$$

$$\text{or, } x \frac{dv}{dx} = \frac{v}{1 + v^3} - v$$

$$\text{or, } x \frac{dv}{dx} = \frac{v - v - v^4}{1 + v^3}$$

$$\text{or, } x \frac{dv}{dx} = \frac{-v^4}{1 + v^3}$$

$$\text{or, } \frac{1 + v^3}{v^4} dv = -\frac{dx}{x}$$

$$\text{or, } \left(v^{-4} + \frac{1}{v}\right) dv = -\frac{dx}{x}$$

Integrating, we get,

$$\frac{v^{-3}}{-3} + \log v = -\log x + \log c$$

$$\text{or, } \frac{-1}{3} \cdot \frac{x^3}{y^3} = -\left(\log \frac{y}{x} + \log x - \log c\right)$$

$$\text{or, } \frac{x^3}{3y^3} = \log \left(\frac{y}{c}\right)$$

$$\therefore x^3 = 3y^3 \log \left(\frac{y}{c}\right).$$

Example 5. Solve: $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$.

Solution

The given equation is

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$$

Put $y = vx$.

$$\text{Then } \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Now, the given equation is

$$v + x \frac{dv}{dx} = \frac{x^2 + v^2 x^2}{2x^2}$$

$$\text{or } v + x \frac{dv}{dx} = \frac{1 + v^2}{2}$$

$$\text{or, } x \frac{dv}{dx} = \frac{1 + v^2}{2} - v$$

$$\text{or, } x \frac{dv}{dx} = \frac{v^2 - 2v + 1}{2}$$

$$\text{or, } x \frac{dv}{dx} = \frac{(v-1)^2}{2}$$

$$\text{or, } \frac{2dv}{(v-1)^2} = \frac{dx}{x}$$

Integrating, we have,

$$\frac{-2}{v-1} = \log x + \log c$$

$$\text{or, } \frac{-2}{\frac{y}{x} - 1} = \log cx$$

$$\text{or, } \frac{2x}{x-y} = \log cx$$

$$\therefore 2x = (x-y) \log cx.$$

Example 6. Solve: $2 \frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

Solution

$$\text{Given, } 2 \frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$$

Put $y = vx$.

$$\text{Then, } \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Now, given equation can be written as

$$2 \left(v + x \frac{dv}{dx} \right) = \frac{vx}{x} + \frac{v^2 x^2}{x^2}$$

$$\text{or, } 2v + 2x \frac{dv}{dx} = v + v^2$$

$$\text{or, } 2x \frac{dv}{dx} = v^2 - v$$

$$\text{or, } \frac{2}{v(v-1)} dv = \frac{dx}{x}$$

$$\text{or, } 2 \left(\frac{1}{v-1} - \frac{1}{v} \right) dv = \frac{dx}{x}$$

Integrating, we get,

$$2 \left[\int \frac{1}{v-1} dv - \int \frac{1}{v} dv \right] = \int \frac{dx}{x} + \log c$$

$$\text{or, } 2 \{ \log(v-1) - \log v \} = \log x + \log c$$

$$\text{or, } 2 \log \left(\frac{v-1}{v} \right) = \log cx$$

$$\text{or, } \log \left(\frac{v-1}{v} \right)^2 = \log cx$$

$$\text{or, } \left(\frac{v-1}{v} \right)^2 = cx$$

$$\text{or, } \left(\frac{\frac{y}{x} - 1}{\frac{y}{x}} \right)^2 = cx$$

$$\text{or, } \left(\frac{y-x}{y} \right)^2 = cx$$

$$\therefore (y-x)^2 = cxy^2.$$

Exercise 5.2

Solve the following differential equations:

- $\frac{dy}{dx} = \frac{x+y}{x}$
- $\frac{dy}{dx} = \frac{2x+y}{x}$
- $\frac{dy}{dx} = \frac{2y-x}{x}$
- $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$
- $2xy \frac{dy}{dx} = x^2 + y^2$
- $xy \frac{dy}{dx} = x^2 - y^2$
- $\frac{dy}{dx} = \frac{x+y}{x-y}$
- $\frac{dy}{dx} = \frac{y}{x} - \sin^2 \frac{y}{x}$

Very Short Answer Questions

- Define homogeneous differential equation.
- Write an example of homogeneous differential equation.
- Write the variable separated form of the homogeneous differential equation $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$ by putting $y = vx$.

Answers

- $y = x \log cx$
- $y = x \log cx^2$
- $y - x = cx^2$
- $x^2 = 2y^2 \log cy$
- $y^2 - x^2 = cx$
- $x^2(x^2 - 2y^2) + c = 0$
- $\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2} \log(x^2 + y^2) + c$
- $\cot\left(\frac{y}{x}\right) = \log x + c$

5.7 Exact Differential Equations

A differential equation $M(x, y) dx + N(x, y) dy = 0$ is said to be exact if there exists a function $f(x, y)$ such that $M(x, y) dx + N(x, y) dy = df(x, y)$. That is, the given differential equation is exact if $M(x, y) dx + N(x, y) dy$ is exact or perfect differential.

The differential equation $M dx + N dy = 0$ will be exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ where $\frac{\partial}{\partial x}$ denotes the partial derivative.

NOTE Every differential equation $M(x, y)dx + N(x, y)dy = 0$ is not exact.

For example: $y dx + x dy = 0$... (i) is an exact differential equation since it can be written as $y dx + x dy = d(xy)$.

So, equation (i) can be written as

$d(xy) = 0$
Integrating, we have,
 $xy = c$.

Sometimes, a differential equation which is not exact can be made exact by multiplying both sides by some suitable function of x and y . This suitable function of x and y is called an **integrating factor** of the differential equation.

For example: The differential equation $x dy - y dx = 0$ is not exact.

If we multiply both sides by $\frac{1}{y^2}$ then it becomes exact.

That is $\frac{x dy - y dx}{y^2} = d\left(\frac{x}{y}\right)$

So, given equation becomes

$$d\left(\frac{x}{y}\right) = 0$$

Integrating, we have,

$$\frac{x}{y} = c$$

$$\therefore x = cy.$$

Here, $\frac{1}{y^2}$ is the integrating factor (I.F.)

NOTE A differential equation can have more than one integrating factor (I.F.)

Solution by Inspection

The following types of differentials are very important and are frequently used in solving equations.

- $x dy + y dx = d(xy)$
- $\frac{x dy - y dx}{x^2} = d\left(\frac{y}{x}\right)$
- $\frac{y dx - x dy}{y^2} = d\left(\frac{x}{y}\right)$
- $x dx + y dy = d\left(\frac{x^2}{2}\right) + d\left(\frac{y^2}{2}\right)$
 $= d\left(\frac{x^2 + y^2}{2}\right)$

$$5. \frac{x dy - y dx}{xy} = \frac{dy}{y} - \frac{dx}{x}$$

$$= d(\log y) - d(\log x)$$

$$= d(\log y - \log x)$$

$$= d\left(\log \frac{y}{x}\right)$$

$$6. \frac{2xy dx - x^2 dy}{y^2} = d\left(\frac{x^2}{y}\right)$$

$$7. \frac{2xy dy - y^2 dx}{x^2} = d\left(\frac{y^2}{x}\right)$$

$$8. \frac{y dx - x dy}{x^2 + y^2} = \frac{\frac{y dx - x dy}{x^2}}{1 + \left(\frac{y}{x}\right)^2} = d\left(\tan^{-1} \frac{y}{x}\right)$$

Illustrative Examples

Example 1. Solve: $x dx - y dy = 0$

Solution

$$x dx - y dy = 0$$

$$\text{or, } d\left(\frac{x^2}{2}\right) - d\left(\frac{y^2}{2}\right) = 0$$

$$\text{or, } d\left(\frac{x^2 - y^2}{2}\right) = 0$$

Integrating, we get,

$$\frac{x^2 - y^2}{2} = \frac{c}{2}$$

$$\therefore x^2 - y^2 = c.$$

Example 2. Solve: $x dy + (x + y) dx = 0$.

Solution

$$x dy + (x + y) dx = 0$$

$$\text{or, } x dy + x dx + y dx = 0$$

$$\text{or, } (x dy + y dx) + x dx = 0$$

$$\text{or, } d(xy) + d\left(\frac{x^2}{2}\right) = 0$$

$$\text{or, } d\left(xy + \frac{x^2}{2}\right) = 0$$

Integrating, we get,

$$xy + \frac{x^2}{2} = c.$$

Example 3. Solve: $(x + 2y - 3) dy - (2x - y + 1) dx = 0$.

Solution

Here,

$$(x + 2y - 3) dy - (2x - y + 1) dx = 0$$

$$\text{or, } x dy + 2y dy - 3 dy - 2x dx + y dx - dx = 0$$

$$\text{or, } (x dy + y dx) + d(y^2) - d(x^2) - d(3y) - d(x) = 0$$

$$\text{or, } d(xy) + d(y^2) - d(x^2) - d(3y) - d(x) = 0$$

$$\text{or, } d(xy + y^2 - x^2 - 3y - x) = 0$$

Integrating, we get,

$$xy + y^2 - x^2 - 3y - x = c.$$

Example 4. Solve: $y dx - x dy = xy dy$.

Solution

Here,

$$y dx - x dy = xy dy$$

Dividing both sides by y^2 , we get,

$$\frac{y dx - x dy}{y^2} = \frac{x}{y} dy$$

$$\text{or, } d\left(\frac{x}{y}\right) = \frac{x}{y} dy$$

$$\text{or, } \frac{d\left(\frac{x}{y}\right)}{\frac{x}{y}} = dy$$

$$\text{or, } d\left(\log \frac{x}{y}\right) = dy$$

$$\text{or, } d\left(\log \frac{x}{y} - y\right) = 0$$

Integrating, we get,

$$\log \left(\frac{x}{y}\right) = y + c.$$

Example 5. Solve: $\sec^2 x dx + \sec^2 y dy = 0$

Solution

$$\sec^2 x dx + \sec^2 y dy = 0$$

$$\text{or, } d(\tan x) + d(\tan y) = 0$$

$$\text{or, } d(\tan x + \tan y) = 0$$

Integrating, we get,

$$\tan x + \tan y = c.$$

Example 6. Solve: $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

Solution

The given equation can be written as
 $\tan y (\sec^2 x \, dx) + \tan x (\sec^2 y \, dy) = 0$
 or, $\tan y \, d(\tan x) + \tan x \, d(\tan y) = 0$
 or, $d(\tan x \tan y) = 0$
 Integrating, we get,
 $\tan x \tan y = c.$

Exercise 5.3

Solve the following differential equations by reducing to exact form.

- $x \, dy + y \, dx = 0$
- $2xy \, dy + y^2 \, dx = 0$
- $y \, dx - x \, dy = 0$
- $2xy \, dx - x^2 \, dy = 0$
- $y \, dx + (x + y) \, dy = 0$
- $(2xy + y^2) \, dy + (y^2 + x) \, dx = 0$
- $\frac{dy}{dx} = \frac{y-x+1}{y-x+5}$
- $(x^2 + 5xy^2) \, dx + (5x^2y + y^2) \, dy = 0$
- $\sin x \cos x \, dx + \sin y \cos y \, dy = 0$

Very Short Answer Questions

- Define exact differential equation.
- Find the solution of differential equation $y \, dx + x \, dy = 0$.
- Find the solution of differential equation $x \, dy - y \, dx = 0$.

Answers

- $xy = c$
- $xy^2 = c$
- $x = cy$
- $x^2 = cy$
- $2xy + y^2 = c$
- $3x^2 + 6xy^2 + 2y^3 = c$
- $x^2 + y^2 - 2xy - 2x + 10y = c$
- $2x^3 + 2y^3 + 15x^2y^2 = c$
- $\sin^2 x + \sin^2 y = c$

Very Short Answer Questions

- $xy = c$
- $y = cx$

5.8 Linear Differential Equations

A differential equation is said to be a linear equation if it can be written in the form

$$\frac{dy}{dx} + Py = Q \quad \dots (i)$$

where P and Q are functions of x (not of y) or constants.

To solve linear differential equation, we multiply both sides by $e^{\int P \, dx}$.
 Then (i) becomes,

$$e^{\int P \, dx} \frac{dy}{dx} + Py \cdot e^{\int P \, dx} = Q e^{\int P \, dx}$$

$$\frac{d}{dx} (y e^{\int P \, dx}) = Q e^{\int P \, dx}$$

Integrating both sides, we get,

$$y \cdot e^{\int P \, dx} = \int Q e^{\int P \, dx} \, dx + c$$

which gives the solution of the differential equation (i).

NOTE The factor $e^{\int P \, dx}$ is called integrating factor. In short, we write integrating factor as I.F.

Illustrative Examples

Example 1. Solve: $\frac{dy}{dx} + \frac{y}{x} = 1$.

Solution

Here,

$$\frac{dy}{dx} + \frac{y}{x} = 1 \quad \dots (i)$$

Comparing it with $\frac{dy}{dx} + Py = Q$, we get,

$$P = \frac{1}{x}, Q = 1$$

$$\begin{aligned} \text{I.F.} &= e^{\int P \, dx} \\ &= e^{\int \frac{1}{x} \, dx} \\ &= e^{\log x} \\ &= x \end{aligned}$$

Multiplying both sides of (i) by x , we get,

$$x \frac{dy}{dx} + y = x$$

or $d(y \cdot x) = x \, dx$

Integrating, we have,

$$xy = \frac{x^2}{2} + c.$$

Example 2. Solve: $\frac{dy}{dx} + y = e^x$.

Solution

Given equation is $\frac{dy}{dx} + y = e^x$... (i)

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get,

$$P = 1, Q = e^x$$

$$\begin{aligned} \text{I.F.} &= e^{\int P dx} \\ &= e^{\int 1 dx} \\ &= e^x \end{aligned}$$

Multiplying both sides of (i) by equation, we have,

$$e^x \frac{dy}{dx} + e^x \cdot y = e^x \cdot e^x$$

$$\text{or, } d(y \cdot e^x) = e^{2x} dx$$

Integrating both sides, we get,

$$y \cdot e^x = \frac{e^{2x}}{2} + c$$

$$\text{or, } y = \frac{e^{2x}}{2e^x} + \frac{c}{e^x}$$

$$\therefore y = \frac{1}{2} e^x + c e^{-x}$$

Example 3. Solve: $\frac{dy}{dx} - \frac{y}{x} = 2x^2$.

Solution

Given differential equation is

$$\frac{dy}{dx} - \frac{y}{x} = 2x^2 \quad \dots (i)$$

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get,

$$P = -\frac{1}{x}, Q = 2x^2$$

Now,

$$\text{I.F.} = e^{\int P dx}$$

$$= e^{\int -\frac{1}{x} dx}$$

$$= e^{-\log x}$$

$$= e^{\log x^{-1}}$$

$$= x^{-1} = \frac{1}{x}$$

Multiplying both sides of (i) by $\frac{1}{x}$, we have,

$$\frac{1}{x} \cdot \frac{dy}{dx} - \frac{1}{x} \cdot \frac{y}{x} = \frac{1}{x} \cdot 2x^2$$

$$\frac{d}{dx} \left(y \cdot \frac{1}{x} \right) = 2x$$

$$\text{or, } d\left(\frac{y}{x}\right) = 2x dx$$

Integrating both sides, we get,

$$\frac{y}{x} = \int 2x dx$$

$$\frac{y}{x} = x^2 + c$$

$$\therefore y = x^3 + cx$$

Example 4. Solve: $\tan x \frac{dy}{dx} + y = \sec x$.

Solution

Here,

$$\tan x \frac{dy}{dx} + y = \sec x$$

Dividing both sides by $\tan x$, we have,

$$\frac{dy}{dx} + \cot x \cdot y = \operatorname{cosec} x \dots (i)$$

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get,

$$P = \cot x, Q = \operatorname{cosec} x$$

$$\text{I.F.} = e^{\int P dx}$$

$$= e^{\int \cot x dx}$$

$$= e^{\log \sin x}$$

$$= \sin x$$

Multiplying both sides of equation (i) by I.F., we get,

$$\sin x \frac{dy}{dx} + \cot x \cdot \sin x \cdot y = \operatorname{cosec} x \cdot \sin x$$

$$\text{or, } \sin x \frac{dy}{dx} + \cos x \cdot y = dx$$

$$\text{or, } d(y \cdot \sin x) = dx$$

Integrating, we have,

$$y \sin x = x + c$$

Example 5. Solve: $\sin x \frac{dy}{dx} + (\cos x)y = \sin x \cos x$.

Solution

Here,

$$\sin x \frac{dy}{dx} + (\cos x)y = \sin x \cos x$$

$$\text{or, } \frac{dy}{dx} + \frac{\cos x}{\sin x} y = \cos x \quad \dots(i)$$

Comparing equation (i) with $\frac{dy}{dx} + Py = Q$, we get,

$$P = \frac{\cos x}{\sin x}, Q = \cos x$$

$$\begin{aligned} \text{I.F.} &= e^{\int P dx} \\ &= e^{\int \frac{\cos x}{\sin x} dx} \\ &= e^{\log \sin x} \\ &= \sin x \end{aligned}$$

Multiplying both sides of (i) by $\sin x$, we have,

$$\sin x \frac{dy}{dx} + \cos x \cdot y = \sin x \cdot \cos x$$

$$\text{or, } d(y \cdot \sin x) = \frac{1}{2} \sin 2x dx$$

Integrating, we have,

$$y \sin x = \int \frac{1}{2} \sin 2x dx + c$$

$$\text{or, } y \sin x = \frac{1}{2} \left(\frac{-\cos 2x}{2} \right) + c$$

$$\therefore y \sin x + \frac{1}{4} \cos 2x = c.$$

Equation Reducible to Linear Form

An equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x alone or constants, is called **Bernoulli's equation**. If we divide both sides by y^n , the equation is reduced to linear form.

Example 6. Reduce the equation $\frac{dy}{dx} + \frac{y}{x} = y^2$ in linear form hence solve it.

Solution

Given equation is

$$\frac{dy}{dx} + \frac{y}{x} = y^2$$

Dividing both sides by y^2 , we have

$$y^{-2} \frac{dy}{dx} + \frac{1}{x} y^{-1} = 1 \quad \dots(i)$$

Put $y^{-1} = z$ then

$$-y^{-2} \frac{dy}{dx} = \frac{dz}{dx}$$

Then equation (i) can be written as

$$\frac{dz}{dx} - \frac{1}{x} z = -1 \quad \dots(ii)$$

which is a linear equation.

$$\text{Here, } P = -\frac{1}{x}, Q = -1$$

$$\begin{aligned} \text{I.F.} &= e^{\int P dx} \\ &= e^{\int -\frac{1}{x} dx} \\ &= e^{-\log x} \\ &= e^{\log x^{-1}} \\ &= \frac{1}{x} \end{aligned}$$

Multiplying equation (i) by $\frac{1}{x}$, we get

$$\frac{1}{x} \frac{dz}{dx} - \frac{1}{x^2} z = -\frac{1}{x}$$

$$\text{or, } d\left(\frac{1}{x} \cdot z\right) = -\frac{1}{x} dx$$

Integrating

$$\frac{z}{x} = -\log x + c$$

$$\text{or, } \left(\frac{1}{x}\right) = c - \log x$$

$$\therefore \frac{1}{xy} = c - \log x.$$

Exercise 5.4

Solve the following linear differential equations.

- $\frac{dy}{dx} + y = 1$
- $\frac{dy}{dx} - y = e^x$
- $\frac{dy}{dx} + \frac{y}{x} = x$
- $x \frac{dy}{dx} + y = x^4$

5. $(1+x^2) \frac{dy}{dx} + 2xy = 4x^2$

7. $\sin x \frac{dy}{dx} + y \cos x = x \sin x$

9. $(1+x^2) \frac{dy}{dx} + y = e^{\tan^{-1}x}$

6. $\frac{dy}{dx} + 2y \tan x = \sin x$

8. $\cos^2 x \frac{dy}{dx} + y = 1$

Very Short Answer Questions

1. Define linear differential equation of first order and first degree.
2. Write the conditions for the differential equation $\frac{dy}{dx} + P \cdot y = Q$ to be linear.
3. Write the integrating factor of the linear differential equation $\frac{dy}{dx} + P \cdot y = Q$, where P and Q are constants or functions of x only.
4. Find the integrating factor of differential equation $\frac{dy}{dx} + \frac{y}{x} = e^x$.
5. Find the integrating factor of differential equation $\frac{dy}{dx} + y \tan x = \sec x$.

Answers

1. $y = 1 + ce^{-x}$ 2. $y = e^x(x+c)$ 3. $xy = \frac{x^3}{3} + c$ 4. $xy = \frac{x^5}{5} + c$

5. $y(1+x^2) = \frac{4x^3}{3} + c$ 6. $y \sec^2 x = \sec x + c$ 7. $y \sin x = \sin x - x \cos x + c$

8. $y = 1 + ce^{-\tan x}$ 9. $y = \frac{1}{2} e^{\tan^{-1}x} + ce^{-\tan^{-1}x}$

Very Short Answer Questions

4. x 5. $\sec x$

5.9 Partial Differential Equations

Many physical problems in science and engineering depend on two or more variables. They usually depend on time t and one or more space variables. In such situation, when they are formulated mathematically, the partial differential equations may arise whereas the simplest physical problems can be modeled by ordinary differential equations. In most problems related to heat transfer, quantum mechanics, electromagnetic theory, etc., we need to use PDE's to model and solve them.

We now give the definition of partial differential equation (PDE).

5.9.1 Partial Differential Equations

An equation involving a dependent variable, several independent variables and partial derivatives of dependent variable with respect to independent variables is called a partial differential equation.

If z is a dependent variable depending on two independent variables x and y then the equation of the form $f(x, y, z, p, q, r, s, t) = 0$ is a partial differential equation, where

$$p = \frac{\partial z}{\partial x} \quad q = \frac{\partial z}{\partial y} \quad r = \frac{\partial^2 z}{\partial x^2} \quad s = \frac{\partial^2 z}{\partial x \partial y} \quad t = \frac{\partial^2 z}{\partial y^2}$$

For example,

$$\begin{aligned} \text{a. } p + q &= z \\ \text{b. } r + 3s + 2t &= x + y \end{aligned}$$

Order and Degree of PDE

The order of a PDE is defined as the order of the highest partial derivative occurring in the PDE.

The degree of a PDE is defined as the order of the highest derivative in the PDE.

These are same as in the case of ordinary differential equations.

For example,

Some examples of order and degree of PDE are

$$\text{a. } \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z \quad \text{First order and first degree}$$

$$\text{b. } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{Second order and first degree}$$

$$\text{c. } \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{\partial z}{\partial y} \quad \text{First order and second degree}$$

$$\text{d. } \left(\frac{\partial^2 z}{\partial x^2}\right)^3 + \left(\frac{\partial^3 z}{\partial y^3}\right)^2 = \frac{\partial z}{\partial x} \quad \text{Third order and second degree}$$

Linear and Non-linear PDE

If the dependent variable and its partial derivatives occur only in the first degree and they are not multiplied then it is called linear PDE. Otherwise it is called non-linear.

5.9.2 Partial Differential Equation of First Order

It z is a dependent variable depending on two independent variables x and y then the equation of the form $f(x, y, z, p, q) = 0$ is called the partial differential equation of first order, when $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

For example,

$$\begin{aligned} \text{a. } & p + q = z \\ \text{b. } & \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0 \end{aligned}$$

5.10 Formation of Partial Differential Equation

The PDE can be formed by any one of the following method.

- Eliminating the arbitrary constants
- Eliminating the arbitrary functions

1. Derivation of PDE by eliminating arbitrary constants.

Let us consider an equation $f(x, y, z, a, b) = 0$... (i),

where, a and b are arbitrary constants. Let z be a function of two independent variables x and y . Now, differentiating (i) partially w.r.t. x and y respectively, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

$$\text{i.e. } \frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} = 0 \quad \dots \text{(ii)}$$

and

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

$$\text{i.e. } \frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} = 0 \quad \dots \text{(iii)}$$

By eliminating two arbitrary constants a and b from (i), (ii) and (iii), we obtain an equation of the form $f(x, y, z, p, q) = 0$, which is the partial differential equation of first order.

If we have more than two constants then the above procedure of elimination will give a PDE.

Example 1. Find the partial differential equation from the relation $z = ax + by + ab$.

Solution

Given,

$$z = ax + by + ab \quad \dots \text{(i)}$$

Differentiating equation (i) partially with respect to x and y respectively, we get

$$\frac{\partial z}{\partial x} = a$$

$$\text{i.e. } p = a$$

and

$$\frac{\partial z}{\partial y} = b$$

$$\text{i.e. } q = b$$

Eliminating a and b from (i), (ii) and (iii), we get

$$z = px + qy + pq$$

which is the required PDE.

Example 2. From the partial differential equation from the relation $(x-a)^2 + (y-b)^2 + z^2 = c^2$

Solution

Given,

$$(x-a)^2 + (y-b)^2 + z^2 = c^2 \quad \dots \text{(i)}$$

Differentiating equation (i) partially with respect to x and y respectively, we get

$$2(x-a) + 2z \frac{\partial z}{\partial x} = 0 \text{ and } 2(y-b) + 2z \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow (x-a) = -z \frac{\partial z}{\partial x} \text{ and } (y-b) = -z \frac{\partial z}{\partial y}$$

Substituting these values in equation (i), we obtain

$$\left(-z \frac{\partial z}{\partial x}\right)^2 + \left(-z \frac{\partial z}{\partial y}\right)^2 + z^2 = c^2$$

$$\text{or, } z^2 \left\{ \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 \right\} = c^2$$

$$\therefore z^2 (p^2 + q^2 + 1) = c^2, \text{ which is the required PDE.}$$

Example 3. Form a partial differential equation if $z = ke^{ax} \sin ay$.

Solution

Given,

$$z = ke^{ax} \sin ay \quad \dots \text{(i)}$$

Differentiating equation (i) partially with respect to x and y respectively, we get

$$\frac{\partial z}{\partial x} = kae^{ax} \sin ay \quad \dots (ii)$$

and

$$\frac{\partial z}{\partial y} = kae^{ax} \cos ay \quad \dots (iii)$$

Again, differentiating equation (ii) partially with respect to x and equation (iii) partially with respect to y , we get,

$$\frac{\partial^2 z}{\partial x^2} = ka^2 e^{ax} \sin ay \quad \dots (iv)$$

and

$$\frac{\partial^2 z}{\partial y^2} = -ka^2 e^{ax} \sin ay \quad \dots (v)$$

Adding equation (iv) and equation (v), we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = ka^2 e^{ax} \sin ay - ka^2 e^{ax} \sin ay$$

$$\text{or, } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

i.e. $r + s = 0$ which is the required PDE.

2. Derivation of PDE by eliminating arbitrary functions.

Suppose u and v are any two given functions of x , y and z . Let f be an arbitrary function of u and v which is written as

$$f(u, v) = 0 \quad \dots (i)$$

We treat x and y as independent variables and z as a dependent variables.

Then,

$$\frac{\partial x}{\partial y} = 0 \text{ and } \frac{\partial y}{\partial x} = 0 \quad \dots (ii)$$

Differentiating (i) partially with respect to x , we get

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial x} \right) = 0$$

$$\text{or, } \frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial x} \right) = 0 \quad [\text{Using (ii)}]$$

$$\text{or, } \frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right) = 0 \quad \left[\because \frac{\partial z}{\partial x} = p \right]$$

$$\text{or, } \frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right) = - \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right)$$

$$\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} = \frac{\left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right)}{\left(\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right)} \quad \dots (iii)$$

In the same way, differentiating (i) partially with respect to y , we get

$$\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} = \frac{\left(\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} \right)}{\left(\frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z} \right)} \quad \dots (iv)$$

Eliminating the arbitrary function f from (iii) and (iv), we obtain

$$\frac{\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z}} = \frac{\frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z}}{\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z}}$$

$$\text{or, } \left(\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} \right) \left(\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right) = \left(\frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z} \right) \left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right)$$

$$\text{or, } \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} + q \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} + p \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} + pq \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial z} = \frac{\partial v}{\partial y} \cdot \frac{\partial u}{\partial x} + p \frac{\partial v}{\partial y} \cdot \frac{\partial u}{\partial z} + q \frac{\partial v}{\partial z} \cdot \frac{\partial u}{\partial x} + pq \frac{\partial v}{\partial z} \cdot \frac{\partial u}{\partial z}$$

$$\text{or, } p \left(\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y} \right) + q \left(\frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z} \right) = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}$$

$$\text{i.e. } Pp + Qq = R \quad \dots (v)$$

where,

$$P = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y}$$

$$Q = \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}$$

$$\text{and } R = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}$$

Equation (v) is the required PDE.

Example 4. Form a partial differential equation by elimination arbitrary function from the relation: $z = f(x^2 + y^2)$

Solution

Given,

$$z = f(x^2 + y^2) \quad \dots (i)$$

Differentiating equation (i) partially with respect to x and y respectively, we get

$$\frac{\partial z}{\partial x} = f'(x^2 + y^2) \cdot 2x$$

$$\Rightarrow f'(x^2 + y^2) = \frac{1}{2x} \frac{\partial z}{\partial x} \quad \dots (ii)$$

and

$$\frac{\partial z}{\partial y} = f'(x^2 + y^2) \cdot 2y$$

$$\Rightarrow f'(x^2 + y^2) = \frac{1}{2y} \frac{\partial z}{\partial y} \quad \dots (iii)$$

From equation (ii) and equation (iii), we get

$$\frac{1}{2x} \frac{\partial z}{\partial x} = \frac{1}{2y} \frac{\partial z}{\partial y}$$

$$\text{or, } x \frac{\partial z}{\partial y} = y \frac{\partial z}{\partial x}$$

$$\text{or, } xq = yp$$

$$\therefore yp - xq = 0$$

Example 5. Form a PDE by eliminating f from $lx + my + nz = f(x^2 + y^2 + z^2)$ **Solution**

Given,

$$lx + my + nz = f(x^2 + y^2 + z^2) \quad \dots (i)$$

Differentiating equation (i) partially with respect to x and y respectively, we get

$$l \cdot 1 + 0 + n \frac{\partial z}{\partial x} = f'(x^2 + y^2 + z^2) \cdot (2x + 2z \cdot \frac{\partial z}{\partial x})$$

$$\text{or, } l + np = f'(x^2 + y^2 + z^2) \cdot (2x + 2zp)$$

$$\Rightarrow f'(x^2 + y^2 + z^2) = \frac{l + np}{2x + 2zp} \quad \dots (ii)$$

and

$$0 + m \cdot 1 + n \cdot \frac{\partial z}{\partial y} = f'(x^2 + y^2 + z^2) \cdot (2y + 2z \cdot \frac{\partial z}{\partial y})$$

$$\text{or, } m + nq = f'(x^2 + y^2 + z^2) \cdot (2y + 2zq)$$

$$\Rightarrow f'(x^2 + y^2 + z^2) = \frac{m + nq}{2y + 2zq} \quad \dots (iii)$$

From equation (ii) and equation (iii), we get

$$\frac{l + np}{2x + 2zp} = \frac{m + nq}{2y + 2zq}$$

$$\text{or, } \frac{l + np}{x + zp} = \frac{m + nq}{y + zq}$$

$$\text{or, } (l + np)(y + zq) = (m + nq)(x + zp)$$

$$\text{or, } ly + lzq + npy + nzpq = mx + mzp + nqx + npqz$$

$$\therefore (ny - mz)p + (lz - nx)q = mx - ly$$

which is the required PDE.

Example 6. Form the PDE by eliminating the arbitrary functions from $z = f_1(y + 2x) + f_2(y - 2x)$ **Solution**

Given,

$$z = f_1(y + 2x) + f_2(y - 2x) \quad \dots (i)$$

Differentiating equation (i) partially with respect to x and y respectively, we get

$$\frac{\partial z}{\partial x} = f_1'(y + 2x) \cdot 2 + f_2'(y - 2x) \cdot (-2)$$

$$= 2f_1'(y + 2x) - 2f_2'(y - 2x) \quad \dots (ii)$$

and

$$\frac{\partial z}{\partial y} = f_1'(y + 2x) \cdot 1 + f_2'(y - 2x) \cdot 1$$

$$= f_1'(y + 2x) + f_2'(y - 2x) \quad \dots (iii)$$

Differentiating equation (ii) partially with respect to x and differentiating equation (iii) partially with respect to y , we get

$$\frac{\partial^2 z}{\partial x^2} = 4f_1''(y + 2x) + 4f_2''(y - 2x)$$

$$= 4[f_1''(y + 2x) + f_2''(y - 2x)] \quad \dots (iv)$$

and

$$\frac{\partial^2 z}{\partial y^2} = f_1''(y + 2x) + f_2''(y - 2x) \quad \dots (v)$$

From equation (iv) and equation (v), we get

$$\frac{\partial^2 z}{\partial x^2} = 4 \cdot \frac{\partial^2 z}{\partial y^2}$$

i.e. $r = 4t$ which is the required PDE.**Example 7.** Find the partial differential equation of a plane cutting off equal intercepts from the axis of x and y .**Solution**

The equation of plane is

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1 \quad \dots (i) \quad [\because x \text{ and } y \text{ intercepts are equal}]$$

Differentiating (i) partially w.r.t. x and y respectively, we get

$$\frac{1}{a} + \frac{1}{c} \frac{\partial z}{\partial x} = 0$$

$$\text{or, } \frac{1}{a} = -\frac{1}{c} \frac{\partial z}{\partial x} \quad \dots (ii)$$

and

$$\frac{1}{a} + \frac{1}{c} \frac{\partial z}{\partial y} = 0$$

$$\text{or, } \frac{1}{a} = -\frac{1}{c} \frac{\partial z}{\partial y} \quad \dots (iii)$$

From (ii) and (iii), we get

$$-\frac{1}{c} \frac{\partial z}{\partial x} = -\frac{1}{c} \frac{\partial z}{\partial y}$$

$$\therefore p = q$$

Example 8. Form the partial differential equation of the set of all spheres whose centre lie on the axis of z .

Solution

The equation of sphere whose centres line on z -axis is of the form

$$x^2 + y^2 + (z - k)^2 = r^2 \quad \dots(i)$$

where k is any constant and r be the radius of the sphere.

Differentiating (i) partially w.r.t. x and y , we get

$$2x + 2(z - k) \frac{\partial z}{\partial x} = 0$$

$$\text{or, } (z - k) \frac{\partial z}{\partial x} = -x$$

$$\text{or, } -\frac{1}{x} \frac{\partial z}{\partial x} = \frac{1}{z - k} \quad \dots(ii)$$

And,

$$2y + 2(z - k) \frac{\partial z}{\partial y} = 0$$

$$\text{or, } (z - k) \frac{\partial z}{\partial y} = -y$$

$$\text{or, } -\frac{1}{y} \frac{\partial z}{\partial y} = \frac{1}{z - k} \quad \dots(iii)$$

From (ii) and (iii), we get

$$-\frac{1}{x} \frac{\partial z}{\partial x} = -\frac{1}{y} \frac{\partial z}{\partial y}$$

$$\text{or, } yp = xq$$

$$\therefore xq - yp = 0$$

Exercise 5.5

1. Form the partial differential equations by eliminating arbitrary constants from the following relations.

a. $z = a(x + y) + b$

b. $z = (x + a)(y + b)$

c. $z = (x^2 + a)(y^2 + b)$

d. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Form the partial differential equations by eliminating arbitrary functions from the following relations.

a. $z = \phi(x^2 - y^2)$

b. $z = \phi(x^2 + y^2)$

c. $z = f(xy)$

d. $z = f\left(\frac{x}{y}\right)$

e. $z = y^2 + 2f\left(\frac{1}{x} + \ln y\right)$

f. $x + y + z = f(x^2 + y^2 + z^2)$

g. $z = \phi(x + ay) + \psi(x - ay)$ h. $z = f_1(x + iy) + f_2(x - iy)$

Answers

1. a. $p = q$

b. $z = pq$

c. $pq = 4xyz$

d. $xp^2 - pz + xzr = 0$

2. a. $qx + py = 0$

b. $xq - yp = 0$

c. $px - qy = 0$

d. $px + qy = 0$

e. $x^2p + yq = 2y^2$

f. $p(y - z) + q(z - x) = x - y$

g. $t = a^2r$

h. $r + t = 0$

5.11 Solutions of Linear PDEs of First Order

In this section, we discuss the solution of linear PDEs of first order. According to the syllabus, our study will be limited to study linear PDEs of first order and its solution.

The solution of a PDE means the relation obtained by reducing the equation into the relation free of derivatives. While making the relation free from the derivative terms, we use several constants and arbitrary functions according to the nature of PDE. If initial or boundary conditions are given then we evaluate the values of the constants and arbitrary functions.

Different types of solutions of PDEs are given below:

Let

$$f(x, y, z, p, q) = 0$$

... (i) be a given PDE.

- Complete integral:** The solution of equation (i) in the form $F(x, y, z, a, b) = 0$, where a and b are arbitrary constants is called complete integral of (i).
- General integral:** The solution of equation (i) in the form $F(x, y, z, a, \phi(a)) = 0$, where a and $\phi(a)$ are constants is called general integral of (i).
- Particular integral:** If we find the values of a and b in the complete integral $F(x, y, z, a, b) = 0$ by using the given conditions, the solution so obtained is called particular integral of (i).
- Singular integral:** If we eliminate a and b in the complete integral $F(x, y, z, a, b) = 0$ then the solution so obtained is called singular integral of (i).

A linear partial differential equation of first order is of the form

$$Pp + Qq = R \quad \dots(i)$$

where P, Q, R are functions of x, y, z and $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$. The equation (i) is called **Lagrange's partial differential equation**.

From previous section, by eliminating the arbitrary function f from

$$f(u, v) = 0 \quad \dots(ii)$$

the PDE $Pp + Qq = R$ has been obtained.

Thus (ii) is the general integral of (i). Here u and v are functions of x, y and z .

So, let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$

where c_1 and c_2 are arbitrary constants.

By differentiation, we get

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0 \quad \dots(iii)$$

and

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0 \quad \dots(iv)$$

Solving equations (iii) and (iv) for dx, dy and dz , we get

$$\frac{dx}{\frac{\partial u}{\partial y} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}}$$

$$\text{or, } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \dots(v)$$

where

$$P = \frac{\partial u}{\partial y} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial y}$$

$$Q = \frac{\partial u}{\partial z} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z}$$

$$R = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}$$

The equation (v) is called Lagrange's auxiliary or subsidiary or characteristics equations for $Pp + Qq = R$.

Hence, if $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two solutions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ then $f(u, v) = 0$ is the general solution.

Steps for Solving the PDE $Pp + Qq = R$ by Lagrange's Method

Write the linear PDE of first order in the form $Pp + Qq = R$.

Write Lagrange's auxiliary equations in the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

Solve the equations given in step (2) for two independent solutions. Name two solutions as $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$.

The general solution can be written in any one of the following three equivalent forms.

$f(u, v) = 0$ or $u = f(v)$ or $v = f(u)$ where f is arbitrary function.

There are several methods of solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Briefly, we discuss the following methods.

Method of grouping: We take any two ratios so that it is easily integrable. The two integrals obtained in this way form the complete solution.

Method of Lagrange's multipliers: We write

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx + m dy + n dz}{lP + mQ + nR}$$

Here, l, m, n are Lagrange's multipliers and they are so chosen that $lP + mQ + nR = 0$. The two integrals obtained from this way or obtained one from method (i) and another from method (ii) form the complete solution.

Geometrical Meaning

Let $f(x, y, z) = 0$ be an integral surface generated by the PDE $Pp + Qq = R$.

$$\text{i.e. } Pp + Qq + R(-1) = 0$$

The direction cosines of the normal at any point on the surface $f(x, y, z) = 0$

are proportional to $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$.

Now,

$$\frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z} = \frac{\partial z}{\partial x} : \frac{\partial z}{\partial y} : -1 = p : q : -1$$

Hence, the normal to the surface $f(x, y, z) = 0$ is perpendicular to a line whose direction ratios are P, Q, R . Therefore, $Pp + Qq = R$ and $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

give the same set of surfaces. Thus these equations are equivalent.

Illustrative Examples

Example 1. Solve: $p + q = x$.

Solution

Given,

$$p + q = x \quad \dots(i)$$

Comparing equation (i) with $Pp + Qq = R$, we get

$$P = 1, Q = 1, R = x$$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{or, } \frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x}$$

Taking first two ratios, we get

$$\frac{dx}{1} = \frac{dy}{1}$$

Integrating, we get

$$x = y + c_1$$

$$\text{or, } x - y = c_1$$

Again, taking first and third ratios, we get

$$\frac{dx}{1} = \frac{dz}{x}$$

$$\text{or, } x dx = dz$$

Integrating, we get

$$\frac{x^2}{2} = z + c_2$$

$$\text{or, } \frac{x^2}{2} - z = c_2$$

The general solution is $\phi(c_1, c_2) = 0$

$$\text{i.e. } \phi\left(x - y, \frac{x^2}{2} - z\right) = 0$$

Example 2. Solve: $xz \frac{\partial z}{\partial x} + yz \frac{\partial z}{\partial y} = xy$

Solution

Given

Comparing equation (i) with $Pp + Qq = R$, we get

$$P = xz, Q = yz, R = xy$$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} \quad \dots(ii)$$

Taking first two ratios, we get

$$\frac{dx}{xz} = \frac{dy}{yz}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating, we get

$$\ln x = \ln y + \ln c_1$$

$$\ln x - \ln y = \ln c_1$$

$$\text{or, } \ln\left(\frac{x}{y}\right) = \ln c_1$$

$$\frac{x}{y} = c_1$$

...(iii)

Again, taking second and third ratios of (ii), we get

$$\frac{dy}{yz} = \frac{dz}{xy}$$

$$\text{or, } \frac{dy}{yz} = \frac{dz}{c_1 y \cdot y} \quad [\text{Using (iii)}]$$

$$\text{or, } \frac{dy}{yz} = \frac{dz}{c_1 y^2}$$

$$\text{or, } c_1 y dy = z dz$$

Integrating, we get

$$c_1 \frac{y^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$$

$$\text{or, } c_1 y^2 - z^2 = c_2$$

$$\text{or, } \frac{x}{y} \cdot y^2 - z^2 = c_2 \quad [\text{Using (iii)}]$$

$$\therefore xy - z^2 = c_2 \quad \dots(iv)$$

From (iii) and (ii), the required general solution is $\phi(c_1, c_2) = 0$

$$\text{i.e. } \phi\left(\frac{x}{y}, xy - z^2\right) = 0 \text{ where } \phi \text{ is an arbitrary function.}$$

Illustrative Examples

Example 1. Solve: $p + q = x$.

Solution

Given,

$$p + q = x \quad \dots(i)$$

Comparing equation (i) with $Pp + Qq = R$, we get

$$P = 1, Q = 1, R = x$$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{or, } \frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x}$$

Taking first two ratios, we get

$$\frac{dx}{1} = \frac{dy}{1}$$

Integrating, we get

$$x = y + c_1$$

$$\text{or, } x - y = c_1$$

Again, taking first and third ratios, we get

$$\frac{dx}{1} = \frac{dz}{x}$$

$$\text{or, } x dx = dz$$

Integrating, we get

$$\frac{x^2}{2} = z + c_2$$

$$\text{or, } \frac{x^2}{2} - z = c_2$$

The general solution is $\phi(c_1, c_2) = 0$

$$\text{i.e. } \phi\left(x - y, \frac{x^2}{2} - z\right) = 0$$

Example 2. Solve: $xz \frac{\partial z}{\partial x} + yz \frac{\partial z}{\partial y} = xy$

Solution

Given,

$$xz \frac{\partial z}{\partial x} + yz \frac{\partial z}{\partial y} = xy$$

$$\text{or, } xz \cdot p + yz \cdot q = xy \quad \dots(i)$$

Comparing equation (i) with $Pp + Qq = R$, we get

$$P = xz, Q = yz, R = xy$$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} \quad \dots(ii)$$

Taking first two ratios, we get

$$\frac{dx}{xz} = \frac{dy}{yz}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating, we get

$$\ln x = \ln y + \ln c_1$$

$$\ln x - \ln y = \ln c_1$$

$$\text{or, } \ln\left(\frac{x}{y}\right) = \ln c_1$$

$$\therefore \frac{x}{y} = c_1 \quad \dots(iii)$$

Again, taking second and third ratios of (ii), we get

$$\frac{dy}{yz} = \frac{dz}{xy}$$

$$\text{or, } \frac{dy}{yz} = \frac{dz}{c_1 y \cdot y} \quad [\text{Using (iii)}]$$

$$\text{or, } \frac{dy}{yz} = \frac{dz}{c_1 y^2}$$

$$\text{or, } c_1 y dy = z dz$$

Integrating, we get

$$c_1 \frac{y^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$$

$$\text{or, } c_1 y^2 - z^2 = c_2$$

$$\text{or, } \frac{x}{y} \cdot y^2 - z^2 = c_2 \quad [\text{Using (iii)}]$$

$$\therefore xy - z^2 = c_2 \quad \dots(iv)$$

From (iii) and (ii), the required general solution is $\phi(c_1, c_2) = 0$

$$\text{i.e. } \phi\left(\frac{x}{y}, xy - z^2\right) = 0 \text{ where } \phi \text{ is an arbitrary function.}$$

Example 3. Solve: $(y-z)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = z-x$

Solution

Given,

$$(y-z)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = z-x$$

$$\text{or, } (y-z)p + (x-y)q = z-x \quad \dots(i)$$

Comparing equation (i) with $Pp + Qq = R$, we get

$$P = y-z, Q = x-y, R = z-x$$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{or, } \frac{dx}{y-z} = \frac{dy}{x-y} = \frac{dz}{z-x} = k \text{ (suppose)} \quad \dots(ii)$$

Using the multipliers 1, 1, 1 in (ii), we get

$$\frac{dx + dy + dz}{y-z+x-y+z-x} = k$$

$$\text{or, } dx + dy + dz = 0$$

Integrating, we get

$$x + y + z = c_1 \quad \dots(iii)$$

Again, using the multipliers x, y, z in (ii), we get

$$\frac{xdx + ydy + zdz}{x(y-z) + y(x-y) + y(z-x)} = k$$

$$\text{or, } xdx + ydy + zdz = 0$$

Integrating, we get

$$\frac{x^2}{2} + yz = c_2 \quad \dots(iv)$$

The required general solution is $\phi\left(x + y + z, \frac{x^2}{2} + yz\right) = 0$

Example 4. Solve: $\tan x \cdot \frac{\partial z}{\partial x} + \tan y \cdot \frac{\partial z}{\partial y} = \tan z$.

Solution

Given,

$$\tan x \cdot \frac{\partial z}{\partial x} + \tan y \cdot \frac{\partial z}{\partial y} = \tan z \quad \dots(i)$$

Comparing equation (i) with $Pp + Qq = R$, we get

$$P = \tan x, Q = \tan y, R = \tan z$$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$\dots(ii)$

Taking first two ratios, we get

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

Integrating, we get

$$\int \frac{\cos x}{\sin x} dx = \int \frac{\cos y}{\sin y} dy$$

$$\ln \sin x = \ln \sin y + \ln c_1$$

$$\ln \sin x - \ln \sin y = \ln c_1$$

$$\ln \left(\frac{\sin x}{\sin y} \right) = \ln c_1$$

$$\frac{\sin x}{\sin y} = c_1 \quad \dots(iii)$$

Again, taking second and third ratios of (ii), we get

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Integrating, we get

$$\int \frac{\cos y}{\sin y} dy = \int \frac{\cos z}{\sin z} dz$$

$$\ln \sin y = \ln \sin z + \ln c_2$$

$$\ln \sin y - \ln \sin z = \ln c_2$$

$$\ln \left(\frac{\sin y}{\sin z} \right) = \ln c_2$$

$$\frac{\sin y}{\sin z} = c_2 \quad \dots(iv)$$

The required general solution is $\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$.

Example 5. Solve: $(x^2 - yz)\frac{\partial z}{\partial x} + (y^2 - zx)\frac{\partial z}{\partial y} = z^2 - xy$.

Solution

Given,

$$(x^2 - yz)\frac{\partial z}{\partial x} + (y^2 - zx)\frac{\partial z}{\partial y} = z^2 - xy$$

$$\text{i.e. } (x^2 - yz)p + (y^2 - zx)q = z^2 - xy \quad \dots(i)$$

Comparing equation (i) with $Pp + Qq = R$, we get

$$P = x^2 - yz, Q = y^2 - zx, R = z^2 - xy$$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{or, } \frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \quad \dots(ii)$$

The equation (i) can be written as

$$\frac{dx - dy}{x^2 - yz - y^2 + zx} = \frac{dy - dz}{y^2 - zx - z^2 + xy} = \frac{dz - dx}{z^2 - xy - x^2 + yz}$$

$$\text{or, } \frac{dx - dy}{(x+y)(x-y) + z(x-y)} = \frac{dy - dz}{(y+z)(y-z) + x(y-z)} = \frac{dz - dx}{(z+x)(z-x) + y(z-x)}$$

$$\text{or, } \frac{dx - dy}{(x-y)(x+y+z)} = \frac{dy - dz}{(y-z)(x+y+z)} = \frac{dz - dx}{(z-x)(x+y+z)}$$

$$\text{or, } \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z} = \frac{d(z-x)}{z-x} \quad \dots(iii)$$

Taking first two ratios, we get

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating, we get

$$\ln(x-y) = \ln(y-z) + \ln c_1$$

$$\text{or, } \ln(x-y) - \ln(y-z) = \ln c_1$$

$$\text{or, } \ln\left(\frac{x-y}{y-z}\right) = \ln c_1$$

$$\therefore \frac{x-y}{y-z} = c_1 \quad \dots(iv)$$

Again, taking second and third ratios of (ii), we get

$$\frac{d(y-z)}{y-z} = \frac{d(z-x)}{z-x}$$

Integrating, we get

$$\ln(y-z) = \ln(z-x) + \ln c_2$$

$$\text{or, } \ln(y-z) - \ln(z-x) = \ln c_2$$

$$\text{or, } \ln\left(\frac{y-z}{z-x}\right) = \ln c_2$$

$$\therefore \frac{y-z}{z-x} = c_2 \quad \dots(iv)$$

The required general solution is $\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$.

Exercise 5.6

Find the general solution (integral) of the following PDEs.

$$1. \quad px + x = 0$$

$$2. \quad x^2p + q = z^2$$

$$3. \quad \frac{y^2z}{x}p + zxq = y^2$$

$$4. \quad \left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{zx}\right)q = \frac{x-y}{xy}$$

$$5. \quad x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z$$

$$6. \quad (y+z) \frac{\partial z}{\partial x} + (x+z) \frac{\partial z}{\partial y} = x+y$$

$$7. \quad z = a(p+q)$$

$$8. \quad x^2p + y^2q = z^2$$

$$9. \quad xp - yq + x^2 - y^2 = 0$$

$$10. \quad (mz - ny)p + (nx - lz)q = ly - mx$$

$$11. \quad p + q = \sin x$$

Very Short Answer Questions

1. Define partial differential equation.
2. Define general integral of PDE.
3. Define particular integral of PDE.

Answers

1. $\phi(x^2 + z^2, y) = 0$
2. $z = e^{m/x} \phi(x-y)$
3. $\phi\left(\frac{1}{x} + y, \frac{1}{x} - \frac{1}{z}\right) = 0$
4. $\phi\left(\frac{1}{x} - \frac{1}{z}, \frac{1}{x} - \frac{1}{y}\right) = 0$
5. $\phi(x^3 - y^3, x^2 - z^2) = 0$
6. $\phi(x^2 + y^2 + 2z, xy) = 0$
7. $\phi(xyz, x+y+z) = 0$
8. $\phi(x^2 + y^2 + z^2, lx + my + nz) = 0$
9. $\phi\left(\frac{xy}{2}, \frac{x-y}{z}\right) = 0$
10. $z + \cos x = \phi(y-x)$
11. $(x+y+z)(x-y)^2 = \phi\left(\frac{x-y}{x-z}\right)$

Computational Method

UNIT

6

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6.1 Linear Programming Problem (LPP)

The word linear means the mathematical function which gives linear relationship between the variable (i.e. $y = a + bx$) and programming means planning of systematic decision making. So, linear programming is a mathematical optimization technique.

In decision making environment, decision makers or businessman involve allocation of resources to various business activities with the objectives of maximizing profit, revenue sales etc. or minimizing (controlling) cost, loss, time etc. The various theories are available to solve such optimization problem while working with limited resources. Linear programming problem is the common tool used by decision makers for planning the business activities.

For the first time, this method was developed by Russian Mathematician L.V. Kantorovich and later developed in 1947 by George B. Dantzig for the purpose of scheduling the complicated procurement activities of air force of the United States. Nowadays, this method has become one of the most popular mathematical methods in solving wide range of problems related to business, industry, agriculture etc.

6.2 Linear Inequality in One Variable and its Graphical Solution

We have already discussed about the linear inequality in one variable in first semester (Unit 1). An expression of the type $ax + b < c$ or $ax + b \leq c$ or $ax + b > c$ or $ax + b \geq c$, ($a \neq 0$) is called a linear inequality where x is a variable and a, b, c are constants.

For examples: $x \geq 3$, $2x + 1 \leq -1$, $x > -1$ etc. are linear inequalities.

The set of the values of x which satisfy the given inequality is called the solution set of the inequality.

Example 1. Solve the inequality $2x + 1 < 7$.

Solution

$$2x + 1 < 7$$

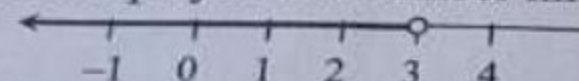
$$\text{or, } 2x < 7 - 1$$

$$\text{or, } 2x < 6$$

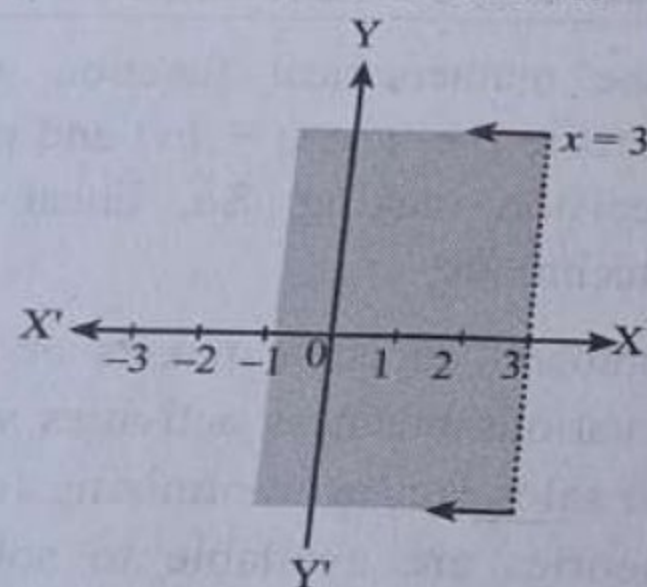
$$\therefore x < 3$$

The solution set in $\{x : x < 3\} = (-\infty, 3)$

The solution set can be displayed in a number line as follows.

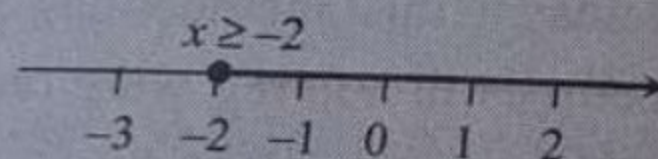


In two dimensions, it can be shown as follows:



Note that $x = 3$ is the boundary line. The boundary line is not included. In this case, the dotted line is used to represent the boundary line.

- NOTE**
- If the inequality contains $<$ or $>$ sign, dotted line is used to represent the boundary line as it is not included in the solution set.
 - If the inequality contains $\leq$ or $\geq$ sign, dark (bold) line is used to represent the boundary line as it is included in the solution set.



6.3 Linear Inequalities in Two variables and their Solution

An expression of the form $ax + by < c$ is called an inequality in two variables x and y ; where a, b, c are constants and a, b are not zero simultaneously. Such expression is also called an inequality in two variables if $<$ is replaced by $>$ or $\leq$ or $\geq$.

Example 2. Graph the half plane given by: $y - x \geq 1$.

The given inequality is $y - x \geq 1$

The boundary line is $y - x = 1$

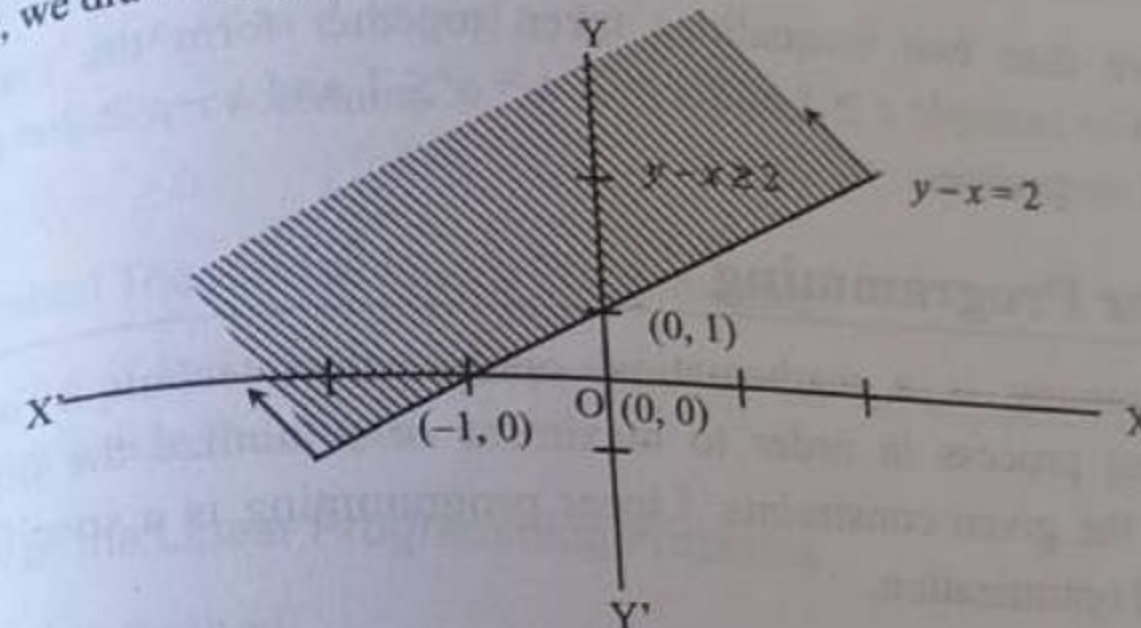
x	0	-1
y	1	0

Taking $(0, 0)$ as a testing point, we get,

$$0 - 0 \geq 1 \text{ (False)}$$

Hence, the half plane determined by the given inequality does not contain origin.

Now, we draw the graph.



Example 3. Determine the half plane given by the inequality $2x - y < 2$.

Solution

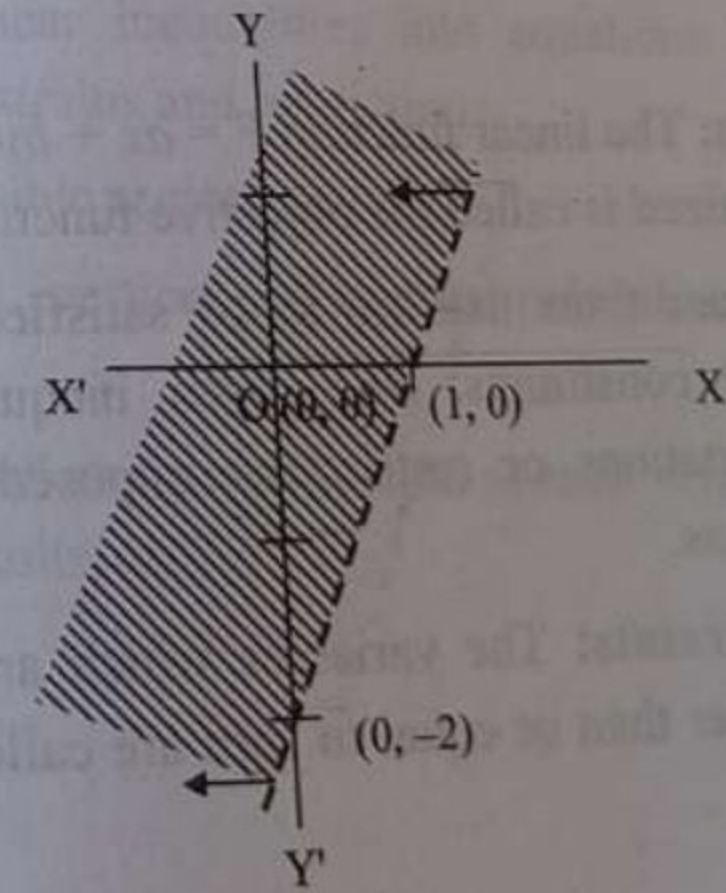
The given inequality is $2x - y < 2$

The corresponding boundary line is $2x - y = 2$

x	0	1
y	-2	0

Take $(0, 0)$ as a testing point, we get

$$2 \times 0 - 0 < 2 \text{ (True)}$$



Hence, the half plane determined by the given inequality is on the same side of testing point $(0, 0)$. But it does not contain the boundary line. So on the graph we draw the dotted boundary line. The half planes are the solutions of the inequalities.

NOTE

- If the line does not pass through $(0, 0)$, then origin is the best testing point.
- The half planes containing the boundary lines are called closed half planes and not containing the boundary lines are called open half planes.

System of Inequalities

Two or more than two inequalities taken together form the system of inequalities. For example $x \geq 1$ and $x \geq 2$; $x + y \leq 1$ and $x - y \geq 2$ etc., form the system of inequalities.

6.4 Linear Programming

Linear programming is a mathematical optimization technique used in decision making process in order to maximize or minimize the objective function under the given constraints. Linear programming is a specific case of mathematical optimization.

Thus, linear programming is a mathematical technique for the optimization (maximization and minimization) of a linear objective function subject to the given linear inequalities or linear equations.

Some Terminologies Used in LPP

- Decision variables:** The quantities that the decision makers want to determine are decision variables.

In order to formulate the linear programming problem, we first have to define a set of decision variables. Decision variables may be men, materials etc.

- Objective functions:** The linear function $F = ax + by + c$ which is to be maximized or minimized is called the objective function.
- Constraints:** The conditions that are to be satisfied by the decision variables are called constraints. The given inequalities are called constraints. The limitations or restrictions imposed on the decision variables are constraints.
- Non-negativity constraints:** The variables which are assumed to be non-negative i.e. greater than or equal to zero are called non-negativity constraints.

Feasible solution: The set of all those possible solutions that satisfy the given constraints is called the feasible solution.

Optimal solution: Any feasible solution which optimizes the objective function is called an optimal solution.

Mathematical Model of Linear Programming Problem

The linear programming problem can be written as

Optimize $(z) = ax_1 + bx_2 + c$ (objective function)

subject to the set of constraints

$$a_{11}x_1 + a_{12}x_2 (< \text{or } \leq \text{or } > \text{or } \geq) b_1$$

$$a_{21}x_1 + a_{22}x_2 (< \text{or } \leq \text{or } > \text{or } \geq) b_2$$

The non-negative constraints

$$x_1 \geq 0, x_2 \geq 0$$

Fundamental Theorem of Linear Programming

It states that if there is a solution, it occurs on the boundary of the feasible region, not inside the region.

Solution of the Linear Programming Problems

- Graphic method
- Simplex method

First, we discuss about the graphic method to solve the linear programming problem. The following steps are to be done in order to solve LPP by graphic method.

- Identify the linear programming problem i.e. identify the decision variables, objective function, constraints and other restrictions.
- Formulate the LPP.
- Convert the linear inequalities into equations and draw graph that includes all constraints and restrictions.
- Identify the feasible region.
- Find the points (vertices) of convex polygonal region of the feasible region.
- Find the value of objective function at each vertex obtained in (5) and interpret the results.

Illustrative Examples

Example 1. Find the solution set of $4x + 3 \geq 2x - 1$.

Solution

We have,

$$4x + 3 \geq 2x - 1$$

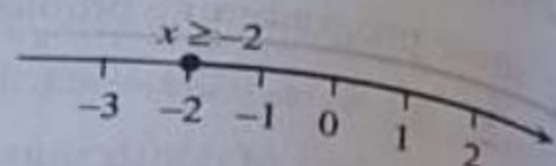
$$\text{or, } 4x - 2x \geq -1 - 3$$

$$\text{or, } 2x \geq -4$$

$$\text{or, } \frac{2x}{2} \geq \frac{-4}{2}$$

$$\text{or, } x \geq -2$$

$\therefore$ The required solution set = $\{x : x \geq -2\}$



Example 2. Determine the solution set of $2x - 1 > 4x + 3$.

Solution

We have,

$$2x - 1 > 4x + 3$$

$$\Rightarrow 2x - 1 + 1 > 4x + 3 + 1$$

$$\Rightarrow 2x > 4x + 4$$

$$\Rightarrow 2x - 4x > 4x - 4x + 4$$

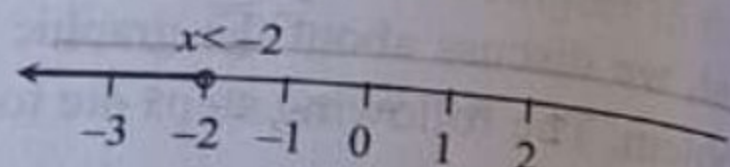
$$\Rightarrow -2x > 4$$

$$\Rightarrow 2x < -4$$

$$\Rightarrow \frac{2x}{2} < \frac{-4}{2}$$

$$\Rightarrow x < -2$$

$\therefore$ Solution set = $\{x : x < -2, x \in \mathbb{R}\}$



Example 3. Draw the graph of the following inequalities: $6x + 5y \leq 30$; $x \geq 1$, $y \geq 2$. Find the feasible region and also find the vertices of the feasible region.

Solution

Given inequalities are

$$6x + 5y \leq 30, x \geq 1, y \geq 2$$

The corresponding equations of the given inequalities are

$$6x + 5y = 30 \quad \dots (i)$$

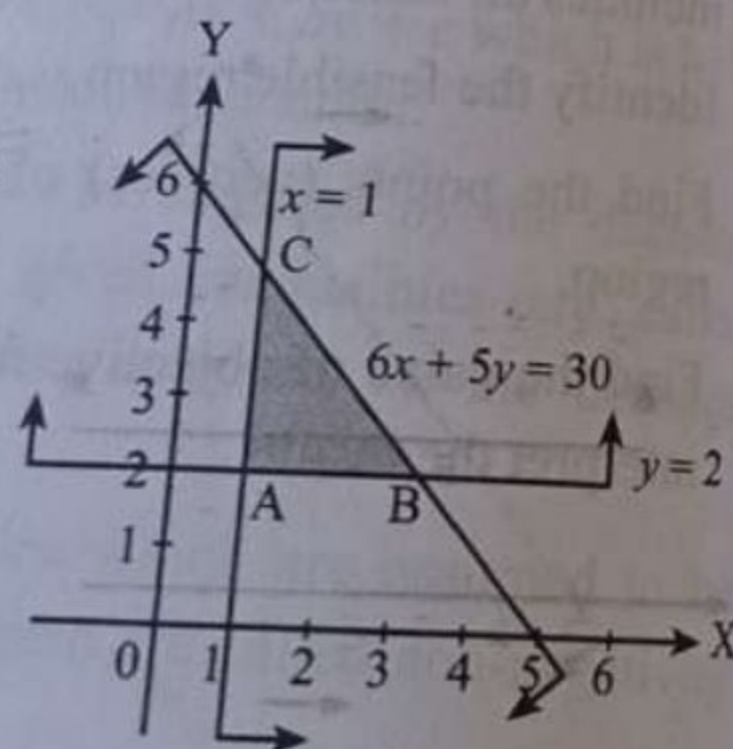
$$x = 1 \quad \dots (ii)$$

$$y = 2 \quad \dots (iii)$$

From (i),

$$6x + 5y = 30$$

When $x = 0$, $y = 6$



When $x = 5$, $y = 0$

The boundary line (i) passes through $(0, 6)$ and $(5, 0)$

Taking testing point $(0, 0)$ i.e. put $x = 0$, $y = 0$ in $6x + 5y \leq 30$
 $0 + 0 \leq 30$ which is true.

Hence the graph of $6x + 5y \leq 30$ is the closed plane region containing the origin.

From (ii) $x = 1$ is a line parallel to y -axis and the region of $x \geq 1$ is the right half plane containing the line $x = 1$.

From (iii), $y = 2$ is a straight line parallel to x -axis

The region of $y \geq 2$ is the upper half plane including the line $y = 2$.

Here ABC is the feasible region and the co-ordinates of vertices of the feasible region are A $(1, 2)$, B $(10/3, 2)$ and C $(1, 24/5)$.

Example 4. Minimize $z = x + y$
subject to the constraints.

$$3x + 2y \geq 12$$

$$x + 3y \geq 9$$

$$x \geq 0, y \geq 0.$$

Solution

Given inequalities are

$$3x + 2y \geq 12, x + 3y \geq 9, x \geq 0, y \geq 0.$$

The corresponding equations of boundary lines are

$$3x + 2y = 12 \quad \dots (i)$$

$$x + 3y = 9 \quad \dots (ii)$$

$$x = 0, y = 0 \quad \dots (iii)$$

From (i) $3x + 2y = 12$

x	0	4
y	6	0

Taking testing point $(0, 0)$ in $3x + 2y \geq 12$, we get, $0 + 0 \geq 12$ (False)

So, the solution set of $3x + 2y \geq 12$ is the closed plane region without containing the origin.

From (ii), $x + 3y = 9$

x	0	9
y	3	0

Taking testing point $(0, 0)$ in $x + 3y \geq 9$, we get, $0 + 0 \geq 9$ (False)

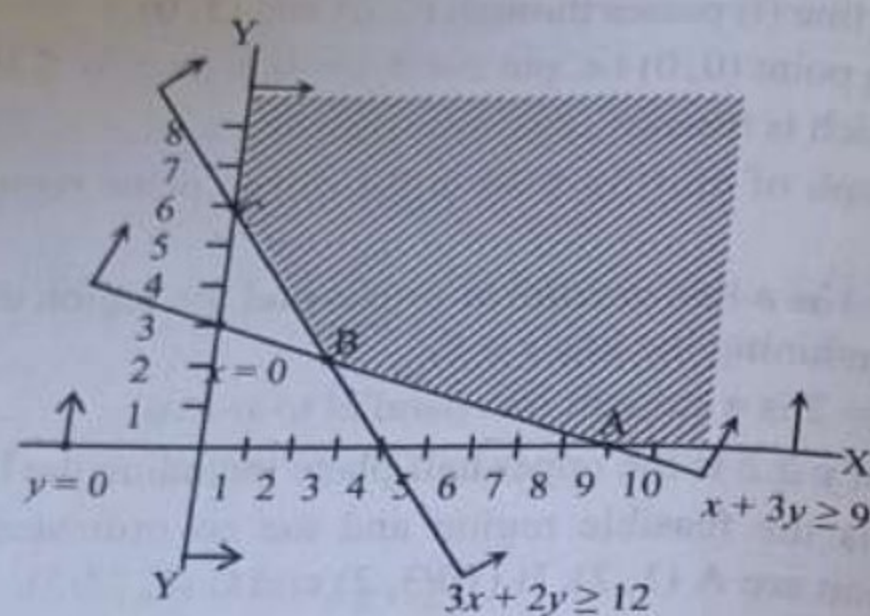
So, the solution set of $x + 3y \geq 9$ is the plane region without containing the origin.

From (iii), $x = 0$ (y -axis)

$y = 0$ (x -axis)

And $x \geq 0, y \geq 0$ means we have to consider the region in first quadrant only.

Graph



The feasible region is shown in the graph. The vertices of the feasible region are $A(9,0)$, $B\left(\frac{18}{7}, \frac{15}{7}\right)$, $C(0,6)$

Vertices	Value of $z = x + y$
$A(9, 0)$	$z = 9 + 0 = 9$
$B\left(\frac{18}{7}, \frac{15}{7}\right)$	$z = \frac{18}{7} + \frac{15}{7} = \frac{33}{7}$
$C(0, 6)$	$z = 0 + 6 = 6$

$\therefore$ Minimum value of $z = \frac{33}{7}$ at $B\left(\frac{18}{7}, \frac{15}{7}\right)$.

Example 5. Find the extreme values of the objective function $F = 16x - 2y + 40$ subject to constraints

$$3x + 4y \leq 24$$

$$0 \leq y \leq 4$$

$$0 \leq x \leq 7.$$

Solution

The boundary equations of given inequalities are

$$3x + 4y = 24 \quad \dots (i)$$

$$y = 0 \quad \dots (ii)$$

$$y = 4 \quad \dots (iii)$$

$$x = 0 \quad \dots (iv)$$

$$x = 7 \quad \dots (v)$$

From (i) $3x + 4y = 24$

x	0	8
y	6	0

Taking testing point $(0, 0)$ in $3x + 4y \leq 24$, we get
 $3 \times 0 + 4 \times 0 \leq 24$ (true)

The graph of $3x + 4y \leq 24$ contains origin.

From (ii) $y = 0$ which is x-axis.

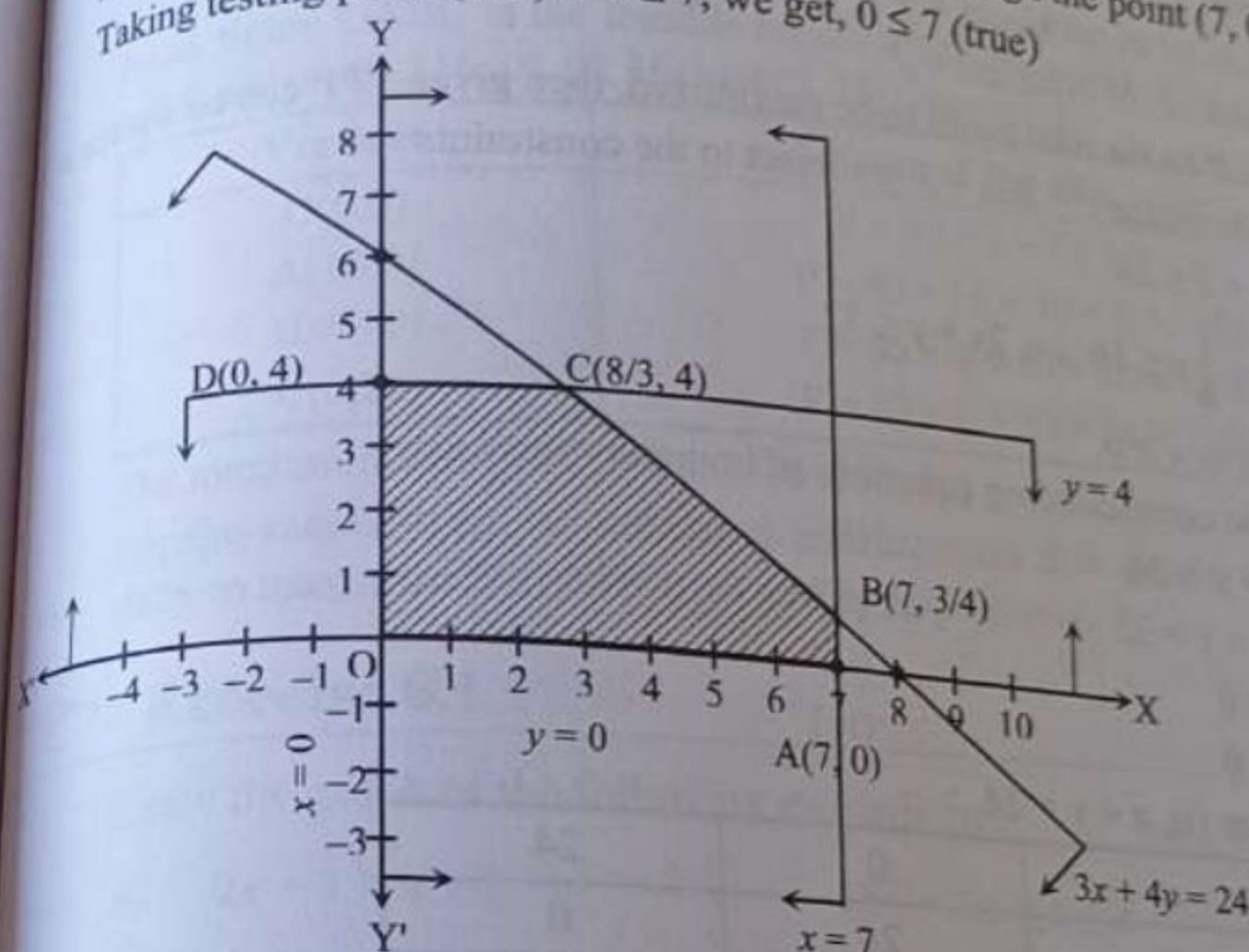
$y \geq 0$ gives the upper half plane containing x-axis.

From (iii) $y = 4$ is the line parallel to x-axis and through the point $(0, 4)$.
 Taking testing point $(0, 0)$ in $y \leq 4$, we get
 $0 \leq 4$ (true)

From (iv) $x = 0$ which is y-axis.

$x \geq 0$ gives the right half plane containing the y-axis.

From (v) $x = 7$ is the line parallel to y-axis and through the point $(7, 0)$.
 Taking testing point $(0, 0)$ in $x \leq 7$, we get, $0 \leq 7$ (true)



The required solution is shaded in the figure.

Vertices	Value of $F = 16x - 2y + 40$
$O(0, 0)$	$F = 16 \times 0 - 2 \times 0 + 40 = 40$
$A(7, 0)$	$F = 16 \times 7 - 2 \times 0 + 40 = 152$
$B\left(7, \frac{3}{4}\right)$	$F = 16 \times 7 - 2 \times \frac{3}{4} + 40 = 150.5$
$C\left(\frac{8}{3}, 4\right)$	$F = 16 \times \frac{8}{3} - 2 \times 4 + 40 = 74.67$
$D(0, 4)$	$F = 16 \times 0 - 2 \times 4 + 40 = 32$

Max. value of $F = 152$ at $(7, 0)$

Min. value of $F = 32$ at $(0, 4)$

Example 6. A small industry manufactures necklaces and bracelets. The combined number of necklace and bracelets that it can handle per day is not more than 24. Each bracelet takes 1 hour of labour to make and each necklace takes a half hour. The total number of hours of labour available does not exceed 16. If the profit on the necklace is Rs. 80 and the profit on the bracelet is Rs. 50. How many of each product should be produced daily to maximize profit?

Solution

Let no of necklaces = x

and no. of bracelets = y

If P be the total profit to be maximized, then given LPP can be written as
Maximize $P = 80x + 50y$ subject to the constraints

$$x + y \leq 24$$

$$x + \frac{1}{2}y \leq 16 \text{ i.e. } 2x + y \leq 32$$

$$x \geq 0, y \geq 0.$$

The corresponding equations of boundary lines are

$$x + y = 24 \quad \dots (i)$$

$$2x + y = 32 \quad \dots (ii)$$

$$x = 0 \quad \dots (iii)$$

$$y = 0 \quad \dots (iv)$$

From (i), $x + y = 24$

x	0	24
y	24	0

Taking testing point $(0, 0)$ in $x + y \leq 24$, we get $0 + 0 \leq 24$ (true)

So, the solution set of $x + y \leq 24$ is the closed plane region containing the origin.

Again, from (ii), $2x + y = 32$

x	0	16
y	32	0

Taking testing point $(0, 0)$ in $2x + y \leq 32$, we get $2 \times 0 + 0 \leq 32$ (true)

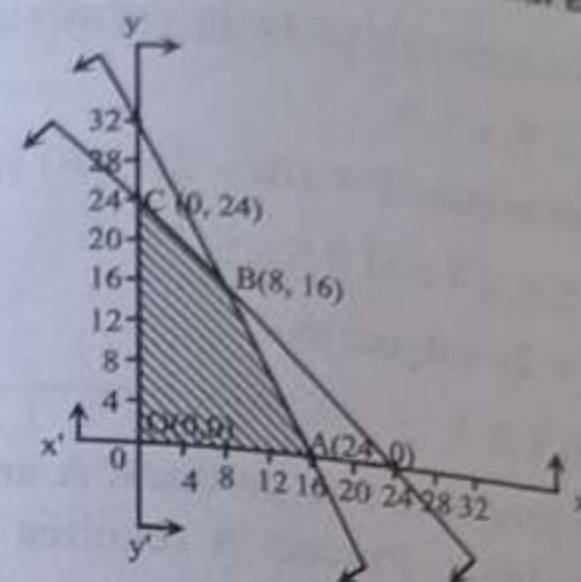
So the solution set of $2x + y \leq 32$ is the closed plane region containing the origin.

From (iii) $x = 0$ which is y -axis.

And $x \geq 0$ gives the closed right half plane.

From (iv) $y = 0$, which is x -axis.

And $y \geq 0$ gives the closed upper half plane.



From figure OABC is the feasible region where the coordinates of O, A, B and C are $(0, 0)$, $(16, 0)$, $(8, 16)$ and $(0, 24)$ respectively.

Vertices	Value of $P = 80x + 50y$
O(0,0)	$P = 80 \times 0 + 50 \times 0 = 0$
A(16,0)	$P = 80 \times 16 + 50 \times 0 = 1280$
B(8,16)	$P = 80 \times 8 + 50 \times 16 = 1440$
C(0,24)	$P = 80 \times 0 + 50 \times 24 = 1200$

The maximum value of P is Rs. 1440 at $(8, 16)$

So, the required no. of necklaces and bracelets that should be produced daily to maximize the profit are 8 and 16 respectively.

Exercise 6.1

- Draw the graph of the following inequalities.
 - $2x + 1 > x + 3$
 - $2x + 1 \leq 3$
 - $3x + y < 6$
 - $4x + y \geq 8$
- Draw the graph of the following system of linear inequalities.
 - $x \geq 5, y \geq 2$
 - $x - y \leq 2, x + y \leq 4$
- Graph the following system of inequalities and find the vertices if they exist.
 - $x + y \leq 6, x \geq 0, y \geq 0$
 - $3x + 4y \leq 24, x \geq 2, y \geq 1$
 - $x + y \leq 6, x - y \geq -2, x \geq 0, y \geq 0$
- Solve the following linear programming problem using graphic method.
 - Maximize $Z = x + y$ subject to the constraints
 $2x + y \leq 14; x + 2y \leq 10, x, y \geq 0$
 - Maximize $Z = 45x + 80y$ subject to
 $x + 2y \leq 7, x - y \leq 4, x \geq 0, y \geq 0$

- c. Minimize $F = 7x + 6y$ subject to the constraints
 $4x - 9y \leq 36, x \geq 9, y \leq 4$
- d. Maximize and minimize $F = 16x - 2y + 40$ subject to constraints
 $3x + 4y \leq 24, 0 \leq y \leq 4$ and $0 \leq x \leq 7$
- e. Optimize $F = x + 2y$ subject to
 $2x + y \leq 7, x \leq y, x \geq 1$

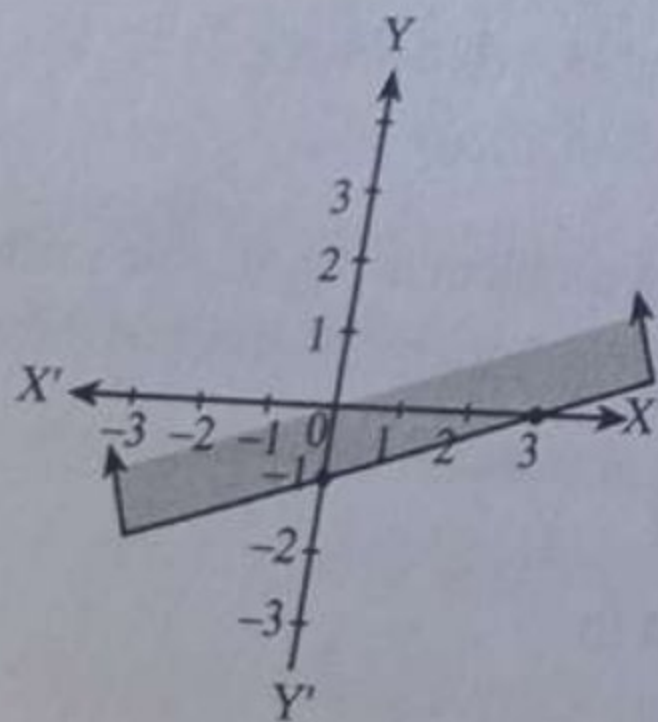
5. An electric company produces two products A and B that are produced and sold on a weekly basis. Product A requires 2 men and B requires only one man as a labour. The company employs a total of 60 workers. The weekly production cannot exceed 25 for product A and 35 for product B because of limited available facilities. Profit margin on A is Rs. 60 and on B is Rs. 40. Formulate it as a linear programming problem and solve for maximum profit using graphic method.
6. Two spare parts X and Y are to be produced in a batch. Each one has to go through two processes P and Q. The time required in hours per unit and total time available are given below:

	X	Y	Total Hours Available
Process P	3	4	24
Process Q	9	4	36

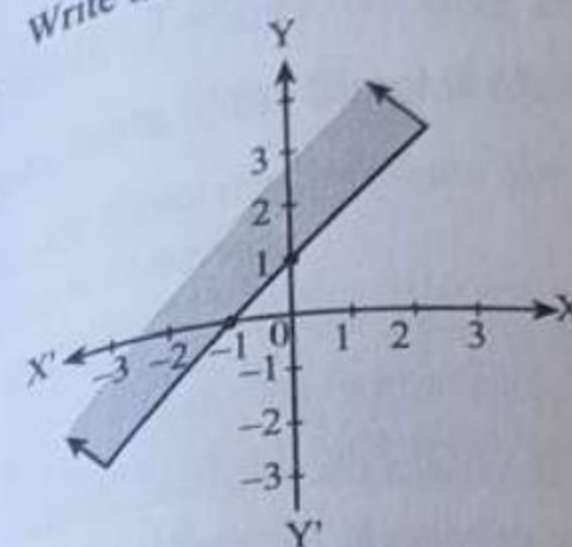
Profit per unit of X and Y are Rs. 5 and Rs. 6 respectively. Find how many number of spare parts of X and Y are to be produced in this batch to maximize the profit.

Very Short Answer Questions

- What does the inequality $x \geq 0$ represent?
- What does the inequality $y \geq 0$ represent?
- Write the inequality representing the following graph?

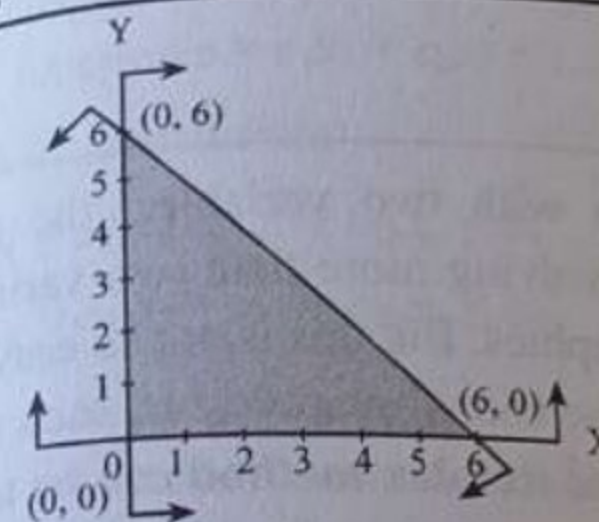


Write the inequality representing the following graph.

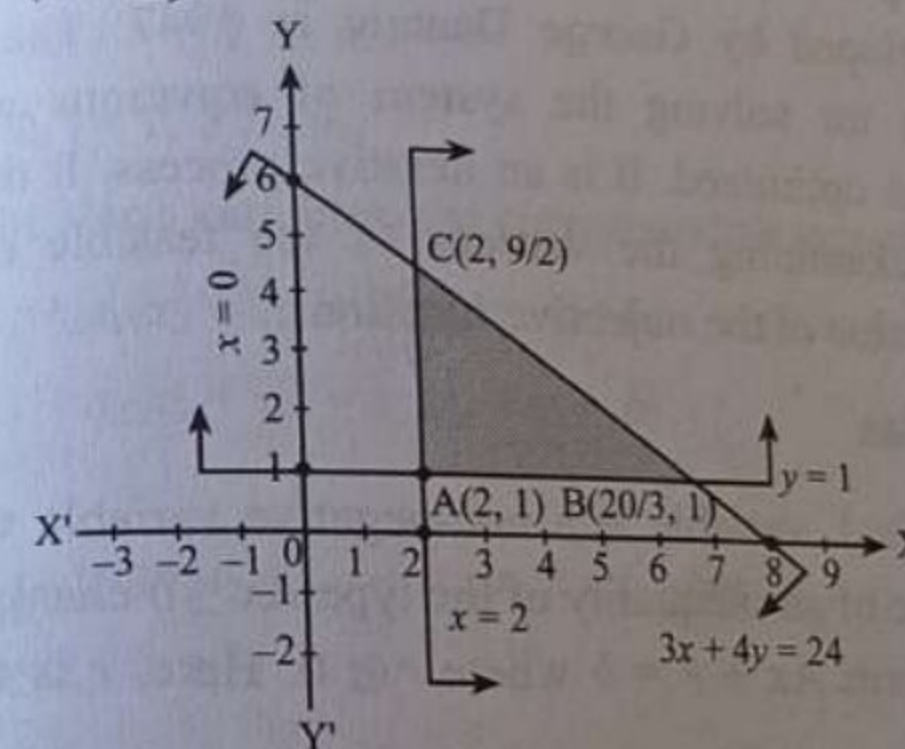


Answers

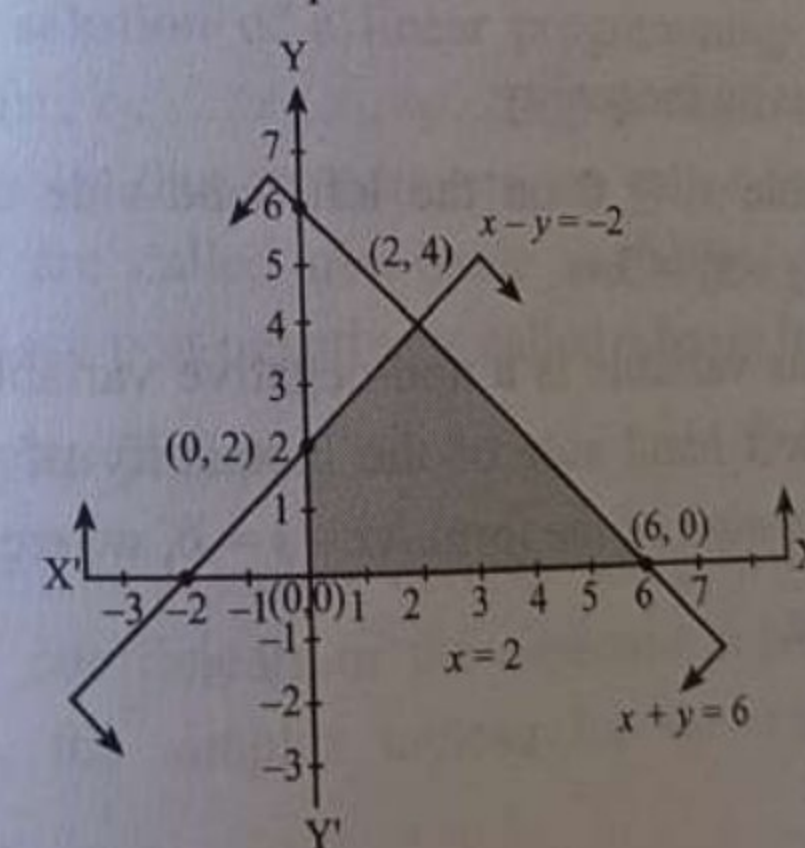
3. a.



b.



c.



4. a. Max. $Z = 8$ at $(6, 2)$
 c. Min. $F = 63$ at $(9, 0)$
 e. Max. $F = 11$ at $(1, 5)$; Min. $F = 3$ at $(1, 1)$
 5. Max. $P = \text{Rs. } (60x + 40y)$
 subject to
 $x \leq 25, y \leq 35$
 $2x + y \leq 60$
 $x \geq 0, y \geq 0$
 Max. $P = \text{Rs. } 2150$ at $x = 12.5, y = 35$
 6. 2, 4.5

Very Short Answer Questions

1. Closed right half plane
2. Closed upper half plane
3. $x - 3y \leq 3$
4. $y - x \geq 1$

6.5 Simplex Method

For the linear programming problems with two variables, the graphical method is convenient. The problems involving more than two variables can be solved by using three dimensional graphics. But this is not so easy because most of us are not skilled with three dimensional graphics. In such situation, the most popular non-graphical procedure **simplex method** can be used. The simplex method is developed by George Dantzig in 1947. The simplex method is an algorithm for solving the system of equations where an objective function is to be optimized. It is an iterative process. It deals with the systematic way of examining the vertices of the feasible region in determining the optimal value of the objective function.

Slack and Surplus Variables

1. **Slack variable:** A slack variable is a non-negative variable which on adding on the left side of an inequality of the type $Ax \leq b$ changes it into an equation of the form $Ax + r = b$ where $r \geq 0$. Here, r is the slack variable.

For example: $x_1 + 2x_2 \leq 3$ is an inequality.

If we add the slack variable $x_3 \geq 0$ on the left hand side of above inequality, we have, $x_1 + 2x_2 + x_3 = 3$.

2. **Surplus variable:** A surplus variable is a non-negative variable which on the subtraction on the left hand side of the inequality of the type $Ax \geq b$ changes it into an equation of the form $Ax - s = b$, where $s \geq 0$.

Here s is the surplus variable.

For example: $2x_1 + 3x_2 \geq 7$.

If we subtract the surplus variable $x_3 \geq 0$ on the left hand side of the above inequality, we have $2x_1 + 3x_2 - x_3 = 7$.

Requirements of the Simplex Method

There are three requirements for solving a linear programming problem by the simplex method.

NOTE

- i. All constraints must be stated as equations.
- ii. The right side of a constraint cannot be negative.
- iii. All the involve variables must be non-negative.

The Simplex Method: Maximization

A linear programming problem in general form if we have to maximize the objective function $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq b_m \end{aligned}$$

where $x_i \geq 0, i = 1, 2, \dots, n$

and $b_i \geq 0, i = 1, 2, \dots, m$

Adding the slack variables, the corresponding equations are

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + s_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + s_2 &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + s_m &= b_m \end{aligned}$$

where $s_i \geq 0, i = 1, 2, \dots, m$

A **basic solution** of a linear programming problem in standard form is a solution $(x_1, x_2, \dots, x_n; s_1, s_2, \dots, s_m)$ in which at most m variables are non zero. The variables that are non zero are called **basic variables**. The variable that are zero are called **non-basic variables**. A basic solution for which all variables are non-negative is called a **basic feasible solution**.

The Simplex Tableau

The augmented matrix corresponding to the constraint equations together with the coefficients of the objective function is the simplex tableau. For example, the simplex tableau for the maximization linear programming problem.

$$z = 10x + 15y \text{ (objective function)}$$

subject to

$$\left. \begin{array}{l} x + 2y \leq 20 \\ x + y \leq 16 \\ x \geq 0, y \geq 0 \end{array} \right\} \text{ (constraints)}$$

Adding the slack variables $r \geq 0, s \geq 0$, we can write the given LPP in standard form

$$x + 2y + r = 20$$

$$x + y + s = 16$$

$$-10x - 15y + z = 0$$

Initial simplex tableau

Basic variables	x	y	r	s	z	RHS (b)
r	1	2	1	0	0	20
s	1	1	0	1	0	16
	-10	-15	0	0	1	0

For this initial simple tableau, the basic variables are r and s ; and non-basic variables which have a value zero are x and y .

The initial basic feasible solution is

$$x = 0, y = 0, r = 20, s = 16, z = 0$$

Optimality check: In order to check optimality of the solution, we have to look at the entries in the bottom row of the simplex tableau. If any one of these entries are negative, then the solution is not optimal. For example, the solution in above example is not optimal as the last row has negative entries. In this case, we improve the current solution.

The improved solution can be obtained with the help of pivoting.

Iteration of Simplex Method

1. The **entering variable** corresponds to the most negative entry in the last row of the simplex tableau.
2. The **departing variable** corresponds to the *smallest non-negative ratio* in the column determined by the entering variable.
3. The entry in the entering variable's column and the departing variable's row is called pivot entry (element).
4. Use elementary row operations so that the pivot entry is 1 and all other entries in that column are zero.

Example 1. Solve the problem
Max. $z = 10x + 15y$
subject to,

$$x + 2y \leq 20$$

$$x + y \leq 16$$

$$x \geq 0, y \geq 0$$

Solution

Step I: Let r and s be the non-negative slack variables. Adding the slack variables, we can write the given LPP in the following form

$$x + 2y + r = 20$$

$$x + y + s = 16$$

$$z = 10x + 15y$$

$$\Rightarrow x + 2y + r + 0 \cdot s + 0 \cdot z = 20$$

$$x + y + 0 \cdot r + s + 0 \cdot z = 16$$

$$-10x - 15y + 0 \cdot r + 0 \cdot s + z = 0$$

Step II: Initial simplex tableau

Basic variables	x	y	r	s	z	RHS (b)
r	1	2	1	0	0	20
s	1	1	0	1	0	16
	-10	-15	0	0	1	0

In this initial simplex tableau, the last row is from objective function. From this tableau, the basic feasible solution is $x = 0, y = 0, r = 20, s = 16$ and $z = 0$. Since the last row contains negative entries, it is not the optimal solution.

Step III: In last (bottom) row, -15 is the most negative entry. So, second column is the pivot column.

Since $\frac{20}{2} = 10, \frac{16}{1} = 16$ and $10 < 16$, so 2 is the pivot element. We have to make pivot element 1 by scaling.

Applying $R_1 \rightarrow \frac{1}{2} R_1$

Basic variables	x	y	r	s	z	RHS (b)
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	10
s	1	1	0	1	0	16
	-10	-15	0	0	1	0

Step IV: Next we make the element in R_2 and R_3 of second column (pivot column) zero by the help of elementary row operations.

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 + 15R_1$

Basic variables	x	y	r	s	z	RHS (b)
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	10
r	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	6
	$-\frac{5}{2}$	0	$\frac{15}{2}$	0	1	150
	$\uparrow$					

Since the last row contains negative entry, so this is not the optimal solution.

Step V: The first column is the pivot column. Since $\frac{10}{\frac{1}{2}} = 20$, $\frac{6}{\frac{1}{2}} = 12$ and

$12 < 20$, so $\frac{1}{2}$ in the second row is pivot element.

Applying $R_2 \rightarrow 2R_2$

Basic variables	x	y	r	s	z	RHS (b)
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	10
x	1	0	-1	2	0	12
	$-\frac{5}{2}$	0	$\frac{15}{2}$	0	1	150

Step VI: Applying $R_1 \rightarrow R_1 - \frac{1}{2}R_2$ and $R_3 \rightarrow R_3 + \frac{5}{2}R_2$

Basic variables	x	y	r	s	z	RHS (b)
y	0	1	1	-1	0	4
x	1	0	-1	2	0	12
	0	0	5	5	1	180

The optimal solution is obtained because all the entries in the last row are non-negative.

The optimal solution is

Max. $z = 180$ at $x = 12$ and $y = 4$.

Summary of the Simplex Method (Maximization)

1. Convert the constraints to equations by adding slack variables.
2. Create the initial simplex tableau.
3. Locate the most negative entry in the last row. The column for this entry is called the **entering column**. If ties occur, we can use any one column for pivot column.
4. The **departing row** corresponds to the smallest non-negative ratio $\frac{b_i}{a_{ij}}$.

If all entries in the entering column are 0 or negative, then there is no maximum value.

For ties, choose either entry. The element in the entering column and departing row is pivot.

Use elementary row operations to make pivot 1 and all other entries in the entering column 0. This process is called pivoting.

If all the entries in the last row are non-negative then the optimal solution is obtained. If not, repeat step 3 to 5 until the entries in the last row are all non-negative.

The Simplex Method (Minimization)

In linear programming problems, there are some problems of maximization and other are of minimization. Every maximization problem is associated with a minimization problem and vice versa. In this case, one problem is called the **dual problem** of other. The original linear programming problem is called a **primal problem** and the derived problem is called a **dual problem**.

A LPP of minimizing the objective function must have the constraints in the form of $\geq$. We solve minimization problem by converting it into maximization problem and use the method stated earlier.

A minimization problem in standard form is

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n \text{ (objective function)}$$

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \geq b_2$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \geq b_m$$

where $x_i \geq 0, i = 1, 2, \dots, n$

$$b_i \geq 0, i = 1, 2, \dots, m$$

Step I: Write the augmented matrix

$$A = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \\ \hline c_1 & c_2 & \dots & c_n & 0 \end{array} \right]$$

Step II: Find the transpose of the matrix A.

$$A^T = \begin{array}{c|cccc} a_{11} & a_{12} & \dots & a_{m1} & c_1 \\ a_{12} & a_{22} & \dots & a_{m2} & c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{1n} & a_2 & \dots & a_{mn} & c_n \\ \hline b_1 & b_2 & \dots & b_m & 0 \end{array}$$

Step III: Form the dual maximization problem as follows

$$\text{Maximize: } w = b_1y_1 + b_2y_2 + \dots + b_my_m$$

subject to

$$a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m \leq c_1$$

$$a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m \leq c_2$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \leq c_n$$

where $y_i \geq 0, i = 1, 2, \dots, m$

Step IV: Apply the simplex method to the dual maximization problem. The maximum value of w will be the minimum value z . The values $x_1, x_2, \dots, x_n$ will occur in last row of the final simplex tableau.

Example 2. Find the dual of the following linear programming problem:

$$\text{Min. } z = 3x + 7y \text{ subject to}$$

$$2x + y \geq 6$$

$$x + y \geq 4$$

$$x, y \geq 0.$$

Solution

$$\text{Here, } A = \left(\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & 1 & 4 \\ 3 & 7 & 0 \end{array} \right) \left. \begin{array}{l} \text{constraints} \\ \text{objective} \end{array} \right\}$$

$$A^T = \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 1 & 7 \\ 6 & 4 & 0 \end{array} \right)$$

The corresponding dual of given linear programming problem is
max. $w = 6r + 4s$

subject to

$$2r + s \leq 3$$

$$r + s \leq 7$$

$$r, s \geq 0.$$

Illustrative Examples

Example 1. Find all the basic solutions to the following system of equations. Which of the solutions are basic feasible solutions?

$$x_1 + x_2 + x_3 = 3$$

$$3x_1 + 5x_2 + x_3 = 12.$$

Solution

There are 3 unknowns and 2 equations. Hence there are two basic variables and one non-basic variable. There are 3 cases.

Case I:

When $x_3 = 0$, we have

$$x_1 + x_2 = 3$$

$$3x_1 + 5x_2 = 12$$

Solving, we get,

$$x_1 = \frac{3}{2}, x_2 = \frac{3}{2} \text{ (basic solutions)}$$

and $x_3 = 0$ (non-basic solution)

Case II:

When $x_2 = 0$, we have

$$x_1 + x_3 = 3$$

$$3x_1 + x_3 = 12$$

Solving, we get,

$$x_1 = \frac{9}{2}, x_3 = -\frac{3}{2} \text{ (basic solutions)}$$

and $x_2 = 0$ (non-basic solution)

Case III:

When $x_1 = 0$, we have

$$x_2 + x_3 = 3$$

$$5x_2 + x_3 = 12$$

Solving, we get,

$$x_2 = \frac{9}{4}, x_3 = \frac{3}{4} \text{ (basic solutions)}$$

and $x_1 = 0$ (non-basic solution)

Since the values in case I and case III are non-negative, so they give the basic feasible solution.

The solutions are:

$$x_1 = \frac{3}{2}, x_2 = \frac{3}{2} \text{ (basic solutions), } x_3 = 0 \text{ (non-basic solution)}$$

$$\text{and } x_2 = \frac{9}{4}, x_3 = \frac{3}{4} \text{ (basic solutions), } x_1 = 0 \text{ (non-basic solution)}$$

Example 2. Using simplex method, maximize $Z = 7x_1 + 5x_2$

subject to:

$$x_1 + 2x_2 \leq 6$$

$$4x_1 + 3x_2 \leq 6$$

$$x_1, x_2 \geq 0.$$

Solution

Let x_3 and x_4 be the non-negative slack variables. Then the given LPP can be written as

$$x_1 + 2x_2 + x_3 = 6$$

$$4x_1 + 3x_2 + x_4 = 6$$

$$7x_1 + 5x_2 = Z$$

$$\Rightarrow x_1 + 2x_2 + x_3 + 0 \cdot x_4 + 0 \cdot Z = 6$$

$$4x_1 + 3x_2 + 0 \cdot x_3 + x_4 + 0 \cdot Z = 6$$

$$-7x_1 - 5x_2 + 0 \cdot x_3 + 0 \cdot x_4 + Z = 0$$

Simplex tableau

Basic variables	x_1	x_2	x_3	x_4	Z	RHS
x_3	1	2	1	0	0	6
x_4	4	3	0	1	0	6
	-7	-5	0	0	1	0
	↑					

Here, -7 is the most negative entry. So first column is the pivot column. Since $\frac{6}{1} = 6$, $\frac{6}{4} = 1.5$ and $1.5 < 6$, so 4 is the pivot element.

Applying $R_2 \rightarrow \frac{1}{4}R_2$

Basic variables	x_1	x_2	x_3	x_4	Z	RHS
x_3	1	2	1	0	0	6
x_1	1	$\frac{3}{4}$	0	$\frac{1}{4}$	0	$\frac{3}{2}$
	-7	-5	0	0	1	0

Applying $R_1 \rightarrow R_1 - R_2$ and $R_3 \rightarrow R_3 + 7R_2$

Basic variables	x_1	x_2	x_3	x_4	Z	RHS
x_3	0	$\frac{5}{4}$	0	$-\frac{1}{4}$	0	$\frac{9}{2}$
x_1	1	$\frac{3}{4}$	0	$\frac{1}{4}$	0	$\frac{3}{2}$
	0	$\frac{1}{4}$	0	$\frac{7}{4}$	1	$\frac{21}{2}$

Since all the entries in the last row are non-negative, so the optimal solution is obtained.

$$\text{Max } Z = \frac{21}{2} \text{ at } x_1 = \frac{3}{2} \text{ and } x_2 = 0.$$

Example 3. Using Simplex method, Maximize $F = 5x - 3y$,
subject to
 $3x + 2y \leq 6$
 $-x + 3y \geq -4$
 $x, y \geq 0.$

Solution
Given inequalities are

$$3x + 2y \leq 6$$

$$-x + 3y \geq -4 \Rightarrow x - 3y \leq 4$$

$$x, y \geq 0$$

Let r and s be the non-negative slack variables. Adding the slack variables, we can write the given LPP in the following form

$$3x + 2y + r = 6$$

$$x - 3y + s = 4,$$

$$5x - 3y = F$$

$$3x + 2y + r + 0 \cdot s + 0 \cdot F = 6$$

$$x - 3y + 0 \cdot r + s + 0 \cdot F = 4$$

$$-5x + 3y + 0 \cdot r + 0 \cdot s + F = 0$$

Simplex tableau

Basic variables	x	y	r	s	F	RHS
r	3	2	1	0	0	6
s	1	-3	0	1	0	4
	-5	3	0	0	1	0
	↑					

Here, -5 is the negative entry in the last row. So, first column is the pivot column. Since $\frac{6}{3} = 2$, $\frac{4}{1} = 4$ and $2 < 4$, so 3 is the pivot element.

Applying $R_1 \rightarrow \frac{1}{3}R_1$

Basic variables	x	y	r	s	F	RHS
x	1	$\frac{2}{3}$	$\frac{1}{3}$	0	0	2
s	1	-3	0	1	0	4
	-5	3	0	0	1	0

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 + 5R_1$

Basic variables	x	y	r	s	F	RHS
x	1	$\frac{2}{3}$	$\frac{1}{3}$	0	0	2
	0	$-\frac{11}{3}$	$-\frac{1}{3}$	1	0	2
s	0	$\frac{19}{3}$	$\frac{5}{3}$	0	1	10

Since all the entries in the last row are non-negative, so the solution is optimal.
Max. value of $F = 10$ at $x = 2, y = 0$.

Example 4. Using Simplex method, Max. $z = 5x_1 + 7x_2$

subject to

$$2x_1 + 3x_2 \leq 13$$

$$3x_1 + 2x_2 \leq 12$$

$$x_1, x_2 \geq 0.$$

Solution

Let x_3 and x_4 be the non-negative slack variables. Then the given LPP can be written as

$$2x_1 + 3x_2 + x_3 = 13$$

$$3x_1 + 2x_2 + x_4 = 12$$

$$5x_1 + 7x_2 = z$$

$$\Rightarrow 2x_1 + 3x_2 + x_3 + 0 \cdot x_4 + 0 \cdot z = 13$$

$$3x_1 + 2x_2 + 0 \cdot x_3 + x_4 + 0 \cdot z = 12$$

$$-5x_1 - 7x_2 + 0 \cdot x_3 + 0 \cdot x_4 + z = 0$$

Simplex tableau

Basic variables	x_1	x_2	x_3	x_4	z	RHS
x_3	2	3	1	0	0	13
x_4	3	2	0	1	0	12
	-5	-7	0	0	1	0

Here, -7 is the most negative entry in the last row. So, second column is the pivot column. Since $\frac{13}{3} = 4.3$, $\frac{12}{2} = 6$ and $4.3 < 6$, so 3 is the pivot element.

Applying $R_1 \rightarrow \frac{1}{3}R_1$

Basic variables	x_1	x_2	x_3	x_4	z	RHS
x_2	$\frac{2}{3}$	1	$\frac{1}{3}$	0	0	$\frac{13}{3}$
x_4	3	2	0	1	0	12
	-5	-7	0	0	1	0

Applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 + 7R_1$

Basic variables	x_1	x_2	x_3	x_4	z	RHS
x_2	$\frac{2}{3}$	1	$\frac{1}{3}$	0	0	$\frac{13}{3}$
x_4	$\frac{5}{3}$	0	$-\frac{2}{3}$	1	0	$\frac{10}{3}$
	$-\frac{1}{3}$	0	$\frac{7}{3}$	0	1	$\frac{91}{3}$

This is not the optimal solution as the last row contains negative entry.

Again, first column is the pivot column. Since, $\frac{13}{\frac{2}{3}} = 6.5$, $\frac{10}{\frac{5}{3}} = 2$ and $2 < 6.5$, so $\frac{5}{3}$ is the pivot element.

Applying $R_2 \rightarrow \frac{3}{5}R_2$

Basic variables	x_1	x_2	x_3	x_4	z	RHS
x_2	$\frac{2}{3}$	1	$\frac{1}{3}$	0	0	$\frac{13}{3}$
x_1	1	0	$-\frac{2}{5}$	$\frac{3}{5}$	0	2
	$-\frac{1}{3}$	0	$\frac{7}{3}$	0	1	$\frac{91}{3}$

Applying $R_1 \rightarrow R_1 - \frac{2}{3}R_2$ and $R_3 \rightarrow R_3 + \frac{1}{3}R_2$

Basic variables	x_1	x_2	x_3	x_4	z	RHS
x_2	0	1	$\frac{3}{5}$	$-\frac{2}{5}$	0	3
x_1	1	0	$-\frac{2}{5}$	$\frac{3}{5}$	0	2
	0	0	$\frac{11}{5}$	$\frac{1}{5}$	1	31

Since all the entries in the last row are non-negative, so the optimal solution is obtained.

Max. $z = 31$ at $x_1 = 2, x_2 = 3$.

Example 5. Using the Simplex method, minimize $W = 3x + 2y$

Subject to

$$2x + y \geq 4$$

$$x + 2y \geq 4$$

$$x, y \geq 0.$$

Solution

A = Augmented matrix formed from the constraints and the objective function

$$= \left(\begin{array}{cc|c} 2 & 1 & 4 \\ 1 & 2 & 4 \\ 3 & 2 & 0 \end{array} \right)$$

$$A^T = \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 2 & 2 \\ 4 & 4 & 0 \end{array} \right)$$

Now, the corresponding dual problem of the given LPP is

$$\text{Max. } Z = 4y_1 + 4y_2$$

$$\text{subject to } 2y_1 + y_2 \leq 3$$

$$y_1 + 2y_2 \leq 2$$

$$y_1, y_2 \geq 0$$

Introducing the slack variables $x_1, x_2 \geq 0$, the problem can be restated as follows:

$$2y_1 + y_2 + x_1 + 0 \cdot x_2 + 0 \cdot z = 3$$

$$y_1 + 2y_2 + 0 \cdot x_1 + x_2 + 0 \cdot z = 2$$

$$-4y_1 - 4y_2 + 0 \cdot x_1 + 0 \cdot x_2 + z = 0$$

Initial Tableau

Basic variables	y_1	y_2	x_1	x_2	z	RHS
x_1	2	1	1	0	0	3
x_2	1	2	0	1	0	2
	-4	-4	0	0	1	0
	↑					

Since $\frac{3}{2} = 1.5$, $\frac{2}{1} = 2$ and $1.5 < 2$, So 2 is the pivot element.

$$\text{Applying } R_1 \rightarrow \frac{1}{2} R_1$$

Basic variables	y_1	y_2	x_1	x_2	z	RHS
y_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{3}{2}$
x_2	1	2	0	1	0	2
	-4	-4	0	0	1	0

$$\text{Applying } R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 + 4R_1$$

Basic variables	y_1	y_2	x_1	x_2	z	RHS
y_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{3}{2}$
x_2	0	$\frac{3}{2}$	$-\frac{1}{2}$	1	0	$\frac{1}{2}$
	0	-2	2	0	1	6
		↑				

Since $\frac{3}{\frac{1}{2}} = 3$, $\frac{1}{\frac{3}{2}} = \frac{1}{3}$ and $\frac{1}{3} < 3$, $\frac{1}{3}$ is pivot.

$$\text{Applying } R_2 \rightarrow \frac{2}{3} R_2$$

Basic variables	y_1	y_2	x_1	x_2	z	RHS
y_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{3}{2}$
y_2	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$
	0	-2	2	0	1	6

$$\text{Applying } R_1 \rightarrow R_1 - \frac{1}{2} R_2, R_3 \rightarrow R_3 + 2R_2$$

Basic variable	y_1	y_2	x_1	x_2	z	RHS
y_1	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{4}{3}$
y_2	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$
	0	0	$\frac{4}{3}$	$\frac{4}{3}$	1	$\frac{20}{3}$

This is the optimal solution as all the entries in last row are non-negative.

$$\text{Max. } Z = \frac{20}{3} \text{ at } y_1 = \frac{4}{3} \text{ and } y_2 = \frac{1}{3}$$

$$\text{i.e. min. } W = \frac{20}{3} \text{ at } x_1 = \frac{4}{3}, x_2 = \frac{4}{3}$$

Example 6. Maximize $z = 10x_1 + 3x_2 + 4x_3$

subject to

$$8x_1 + 2x_2 + 3x_3 \leq 400$$

$$4x_1 + 3x_2 \leq 200$$

$$x_3 \leq 40$$

$$x_1, x_2, x_3 \geq 0.$$

Solution

Let s_1, s_2 and s_3 be the non-negative slack variables. Adding the slack variables, we can write the LPP in the following form:

$$8x_1 + 2x_2 + 3x_3 + s_1 = 400$$

$$4x_1 + 3x_2 + s_2 = 200$$

$$x_3 + s_3 = 40$$

$$10x_1 + 3x_2 + 4x_3 = z$$

$$\Rightarrow 8x_1 + 2x_2 + 3x_3 + s_1 + 0 \cdot s_2 + 0 \cdot s_3 + 0 \cdot z = 400$$

$$4x_1 + 3x_2 + 0 \cdot x_3 + 0 \cdot s_1 + s_2 + 0 \cdot s_3 + 0 \cdot z = 200$$

$$0 \cdot x_1 + 0 \cdot x_2 + x_3 + 0 \cdot s_1 + 0 \cdot s_2 + s_3 + 0 \cdot z = 40$$

$$-10x_1 - 3x_2 - 4x_3 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 + z = 0$$

Initial simplex tableau

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	z	RHS (b)
x_1	8	2	3	1	0	0	0	400
s_2	4	3	0	0	1	0	0	200
s_3	0	0	1	0	0	1	0	40
	-10	-3	-4	0	0	0	1	0

Here, -10 is the most negative entry in the last row. Since $\frac{400}{8} = 50$, $\frac{200}{4} = 50$, let, 8 be the pivot element.

Applying $R_1 \rightarrow \frac{1}{8} R_1$

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	z	RHS (b)
x_1	1	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{8}$	0	0	0	50
s_2	4	3	0	0	1	0	0	200
s_3	0	0	1	0	0	1	0	40
	-10	-3	-4	0	0	0	1	0

Applying $R_2 \rightarrow R_2 - 4R_1$ and $R_4 \rightarrow R_4 + 10R_1$

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	z	RHS (b)
x_1	1	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{8}$	0	0	0	50
s_2	0	2	$-\frac{3}{2}$	$-\frac{1}{2}$	1	0	0	0
s_3	0	0	1	0	0	1	0	40
	0	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{5}{4}$	0	0	1	500

This is not the optimal solution as the last row contains the negative entries. Here, $-\frac{1}{2}$ is the most negative entry in the last row. So 2nd column is the pivot column.

Since, $\frac{50}{\frac{1}{4}} = 200$, $\frac{0}{2} = 0$ and $0 < 200$. So 2 is the pivot.

Applying $R_2 \rightarrow \frac{1}{2} R_2$

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	z	RHS (b)
x_1	1	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{8}$	0	0	0	50
x_2	0	1	$-\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0	0	0
s_3	0	0	1	0	0	1	0	40
	0	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{5}{4}$	0	0	1	500

Applying $R_1 \rightarrow R_1 - \frac{1}{4} R_2$ and $R_4 \rightarrow R_4 + \frac{1}{2} R_2$

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	z	RHS (b)
x_1	1	0	$\frac{9}{16}$	$\frac{3}{16}$	$-\frac{1}{8}$	0	0	50
x_2	0	1	$-\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0	0	0
s_3	0	0	1	0	0	1	0	40
	0	0	$-\frac{5}{8}$	$\frac{9}{8}$	$\frac{1}{4}$	0	1	500

This is not the optimal solution as the last row contains negative entry. Here third column is the pivot column.

Since, $\frac{50}{\frac{9}{16}} = \frac{800}{9}$, $\frac{0}{-\frac{3}{4}} = 0$ (not calculated as divisor is negative)

$\frac{40}{1} = 40$, and $40 < \frac{800}{9}$.

So, 1 is the pivot element.

Applying $R_1 \rightarrow R_1 - \frac{9}{16} R_3$, $R_2 \rightarrow R_2 + \frac{3}{4} R_3$ and $R_4 \rightarrow R_4 + \frac{5}{8} R_3$

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	z	RHS (b)
x_1	1	0	0	$\frac{3}{16}$	$-\frac{1}{8}$	$-\frac{9}{16}$	0	$\frac{55}{2}$
x_2	0	1	0	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	0	30
x_3	0	0	1	0	0	1	0	40
	0	0	0	$\frac{9}{8}$	$\frac{9}{8}$	$\frac{5}{8}$	1	525

Since the last row has non-negative entries, so the optimal solution is obtained.

Max. value of $z = 525$ at $x_1 = \frac{55}{2}$, $x_2 = 30$ and $x_3 = 40$.

Initial simplex tableau

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	z	RHS (b)
x_1	<u>8</u>	2	3	1	0	0	0	400
x_2	4	3	0	0	1	0	0	200
s_3	0	0	1	0	0	1	0	40
	-10	-3	-4	0	0	0	1	0

Here, -10 is the most negative entry in the last row. Since $\frac{400}{8} = 50$, $\frac{200}{4} = 50$, let, 8 be the pivot element.

Applying $R_1 \rightarrow \frac{1}{8} R_1$

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	z	RHS (b)
x_1	1	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{8}$	0	0	0	50
s_2	4	3	0	0	1	0	0	200
s_3	0	0	1	0	0	1	0	40
	-10	-3	-4	0	0	0	1	0

Applying $R_2 \rightarrow R_2 - 4R_1$ and $R_4 \rightarrow R_4 + 10R_1$

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	z	RHS (b)
x_1	1	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{8}$	0	0	0	50
s_2	0	<u>2</u>	$-\frac{3}{2}$	$-\frac{1}{2}$	1	0	0	0
s_3	0	0	1	0	0	1	0	40
	0	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{5}{4}$	0	0	1	500

This is not the optimal solution as the last row contains the negative entries. Here, $-\frac{1}{2}$ is the most negative entry in the last row. So 2nd column is the pivot column.

Since, $\frac{50}{1} = 50$, $\frac{0}{2} = 0$ and $0 < 50$. So 2 is the pivot.

Applying $R_2 \rightarrow \frac{1}{2} R_2$

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	z	RHS (b)
x_1	1	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{8}$	0	0	0	50
x_2	0	1	$-\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0	0	0
s_3	0	0	1	0	0	1	0	40
	0	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{5}{4}$	0	0	1	500

Applying $R_1 \rightarrow R_1 - \frac{1}{4} R_2$ and $R_4 \rightarrow R_4 + \frac{1}{2} R_2$

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	z	RHS (b)
x_1	1	0	$\frac{9}{16}$	$\frac{3}{16}$	$-\frac{1}{8}$	0	0	50
x_2	0	1	$-\frac{3}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0	0	0
s_3	0	0	<u>1</u>	0	0	1	0	40
	0	0	$-\frac{5}{8}$	$\frac{9}{8}$	$\frac{1}{4}$	0	1	500

This is not the optimal solution as the last row contains negative entry. Here third column is the pivot column.

Since, $\frac{50}{9} = \frac{800}{9}$, $\frac{0}{-\frac{3}{4}} = 0$ (not calculated as divisor is negative)

$\frac{40}{1} = 40$, and $40 < \frac{800}{9}$.

So, 1 is the pivot element.

Applying $R_1 \rightarrow R_1 - \frac{9}{16} R_3$, $R_2 \rightarrow R_2 + \frac{3}{4} R_3$ and $R_4 \rightarrow R_4 + \frac{5}{8} R_3$

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	z	RHS (b)
x_1	1	0	0	$\frac{3}{16}$	$-\frac{1}{8}$	$-\frac{9}{16}$	0	$\frac{55}{2}$
x_2	0	1	0	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	0	30
x_3	0	0	1	0	0	1	0	40
	0	0	0	$\frac{9}{8}$	$\frac{9}{8}$	$\frac{5}{8}$	1	525

Since the last row has non-negative entries, so the optimal solution is obtained.

Max. value of $z = 525$ at $x_1 = \frac{55}{2}$, $x_2 = 30$ and $x_3 = 40$.

Exercise 6.2

1. Using simplex method, find the optimal solutions of the following linear programming problems.

a. Max. $Z = 7x_1 + 5x_2$
subject to

$$x_1 + 2x_2 \leq 6$$

$$4x_1 + 3x_2 \leq 12$$

$$x_1, x_2 \geq 0$$

c. Max. $P = x + 3y$
subject to

$$x + y \leq 4$$

$$x - y \leq 1$$

$$x \geq 0, y \geq 0$$

e. Max. $F = x_1 + 7x_2$
subject to

$$x_1 - 2x_2 \geq -8$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

g. Max. $F = 10x + 3y + 4z$
subject to

$$8x + 2y + 3z \leq 400$$

$$4x + 3y \leq 200$$

$$z \leq 40$$

$$x, y, z \geq 0$$

2. Using simplex method, find the optimal solutions of the following linear programming problems.

a. Min. $Z = 10x + 15y$
subject to

$$x + y \geq 8$$

$$5x + 3y \geq 30$$

$$x \geq 0, y \geq 0$$

b. Min. $F = 3x_1 + x_2$
subject to

$$4x_1 + 3x_2 \geq 12$$

$$x_1 + x_2 \geq 5$$

$$x_1, x_2 \geq 0$$

b. Max. $Z = 9x + y$
subject to

$$2x + y \leq 8$$

$$4x + 3y \leq 18$$

$$x \geq 0, y \geq 0$$

d. Max. $P = 50x_1 + 80x_2$
subject to

$$x_1 + 2x_2 \leq 32$$

$$3x_1 + 4x_2 \leq 84$$

$$x_1, x_2 \geq 0$$

f. Max. $Z = 2x_1 + 12x_2 + 8x_3$
subject to

$$2x_1 - 2x_2 + x_3 \leq 100$$

$$x_1 - 2x_2 + 5x_3 \leq 80$$

$$10x_1 + 5x_2 + 4x_3 \leq 80$$

$$x_1, x_2, x_3 \geq 0$$

c. Min. $C = 3x_1 + 2x_2$
subject to

$$-x_1 + x_2 \leq -3$$

$$2x_1 + 5x_2 \geq 10$$

$$x_1, x_2 \geq 0$$

d. Min. $C = 4x + 2y$
subject to

$$x + y \geq 21$$

$$3x + y \geq 27$$

$$x, y \geq 0$$

Very Short Answer Questions

Define surplus variable.

Define slack variable.

What is meant by dual problem?

Define basic variable.

Answers

a. Max. $Z = 21$ at $x_1 = 3, x_2 = 0$

c. Max. $P = 12$ at $x = 0, y = 4$

e. Max. $F = 14$ at $x_1 = 0, x_2 = 2$

g. Max. $F = 525$ at $x = \frac{55}{2}, y = 30, z = 40$

a. Min. $Z = 80$ at $x = 8, y = 0$

c. Min. $C = \frac{83}{7}$ at $x_1 = \frac{25}{7}, x_2 = \frac{4}{7}$

b. Max. $Z = 36$ at $x = 4, y = 0$

d. Max. $P = 1480$ at $x_1 = 20, x_2 = 6$

f. Max. $Z = 192$ at $x_1 = 0, x_2 = 16, x_3 = 0$

b. Min. $F = 5$ at $x_1 = 0, x_2 = 5$

d. Min. $C = 48$ at $x = 3, y = 18$

6.6 System of Linear Equations

An equation of the form $ax + by = c$ is called a linear equation, where x, y are variables and a, b, c are constants.

A system of linear equations is a collection of one or more linear equations involving the same variables.

For example: $a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$ is a system of linear equations.

The system of linear equations in three variables x_1, x_2 and x_3 are

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

where the coefficients of the variables and b_1, b_2, b_3 are real constants.

The system of m linear equations in n variables (unknowns) is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

If $b_i = 0$ for all $i = 1, 2, \dots, m$ then the system of equations is called **homogeneous system** of equations.

A **solution** of the above system is a set $(s_1, s_2, \dots, s_n)$ of a numbers that makes each equation a true statement when the values $s_1, s_2, \dots, s_n$ are substituted for $x_1, x_2, \dots, x_n$.

The set of all possible solutions is called the **solution set** of the linear system.

A system of linear equations has either

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

Definition

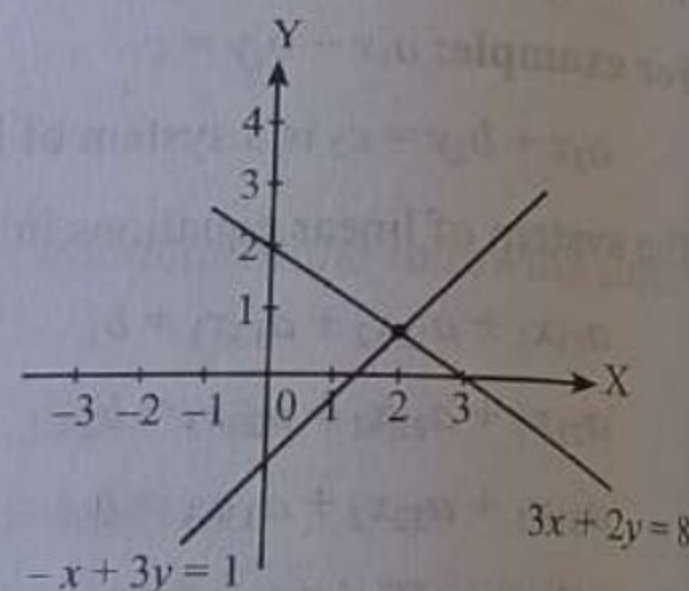
A system of linear equation is said to be **consistent** if it has either one solution or infinitely many solutions and system is said to **inconsistent** if it has no solution.

For example: The system of equations $x_1 + x_2 = 3$ and $x_1 - x_2 = 1$ is consistent since it has a solution namely $x_1 = 2$ and $x_1 = 1$.

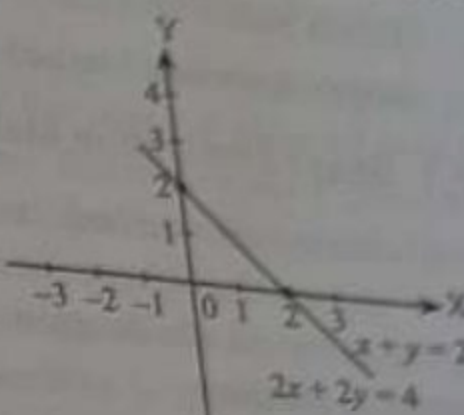
But the system of equation $x_1 + x_2 = 2$ and $2x_1 + 2x_2 = 5$ is inconsistent since it has no solution.

Three different cases are discussed in the following examples:

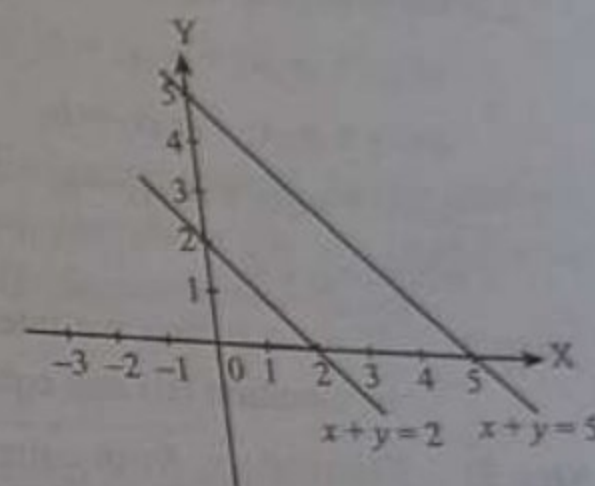
Example 1: If the system has exactly one solution, the system is said to be **consistent and independent**. For example, consider equations $3x + 2y = 8$ and $-x + 3y = 1$. The solution of this system is $x = 2$, $y = 1$. The lines represented by these equations are intersecting and they intersect at $(2, 1)$. Such system is called **consistent and independent**. The graph is shown in the adjoining figure.



Example 2: If the system has an infinite number of solutions, the system is said to be **consistent and dependent**. Consider $x + y = 2$ and $2x + 2y = 4$. Here two equations describe the same line. Both lines have infinite common solutions. The solution for this system is the set of all points on the line. The graph is shown in the adjoining figure.



Example 3: If the system has no solution, the system is said to be **inconsistent and independent**. Consider $x + y = 2$ and $x + y = 5$. From the graph, we see that the two lines do not intersect since the lines are parallel. Thus the system has no solution. The graph is shown in the adjoining figure.



Trivial and Non-trivial Solution

If all the values of the variables in a system of equations is zero then the solution is called **trivial solution**. If at least one value of the variable is non-zero, then the solution is called **non-trivial solution**.

Matrix Notation

Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ x_1 + x_2 &= 2 \\ 3x_1 - 2x_2 + x_3 &= 2 \end{aligned}$$

Then, $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$ is called the **coefficient matrix** or **matrix of coefficients**.

And $\begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 1 & 1 & 0 & : & 2 \\ 3 & -2 & 1 & : & 2 \end{bmatrix}$ is called **augmented matrix** of the system.

6.7 Solution of Linear System

If a system of equations has solution then they can be found by different methods. In this chapter, we discuss the following methods to solve the system of equations.

- a. Gauss Elimination Method

- b. Gauss Seidel Method
c. Matrix Inversion Method

6.7.1 Gauss Elimination Method

Gauss elimination method is the elementary elimination method and it reduces the system of equations to an equivalent **upper-triangular form** using the **forward elimination** approach and then it can be solved by **back substitution method**.

Let the system of equations be

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad \dots (i)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad \dots (ii)$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad \dots (iii)$$

where $a_{11} \neq 0$.

- Step 1:** If $a_{11} = 0$ then we interchange either equation (i) and equation (ii) or equation (ii) and equation (iii) in order to make $a_{11} \neq 0$.
- Step 2:** Eliminate x_1 from equation (i) and equation (ii) to get an equation with variables x_2 and x_3 . Denote this equation by (iv).
- Step 3:** Eliminate x_1 from equation (i) and (iii) to get an equation with variables x_2 and x_3 . Denote this equation by (v).
- Step 4:** Eliminate x_2 and x_3 from equation (iv) and (v) to get an equation with variable x_3 . Denote this equation by (vi).
- Step 5:** Now, we have the following three types of equations:

$$a'_{11}x_1 + a'_{12}x_2 + a'_{13}x_3 = b'_1 \quad \dots (i)$$

$$a'_{22}x_2 + a'_{23}x_3 = b'_2 \quad \dots (iv)$$

$$a'_{33}x_3 = b'_3 \quad \dots (vi)$$

where, $a'_{22}, a'_{23}, a'_{33}, b'_2$ and b'_3 are constants.

This process is known as forward elimination process.

- Step 6:** Obtain x_3 from equation (vi). Use the value of x_3 in equation (iv) to get x_2 . Finally, use x_2 and x_3 in equation (i) to get x_1 . This process is known as **backward substitution**.

The equations involving two variables can also be solved using the above mentioned process but it is shorter.

Pivot Elements

In Gaussian Elimination method, the coefficients a_{ij} of the variables in the equation are called pivot elements, where $i = j$.

Gauss Elimination Method with Partial Pivoting and Complete Pivoting

Now, we discuss the case when the pivot is zero or close to zero. If the pivot is zero, the elimination process fails and if the pivot is close to zero then round off errors may occur. The procedure that avoids such problems is pivoting. If a_{11} is zero or very close to zero then interchange this first row with the row having the largest coefficient in the column below the pivot column. This process is called **partial pivoting**. If we search both columns and rows for the largest element, the procedure is **complete pivoting**. But complete pivoting involves more complexity in computations. Because of this reason, complete pivoting is rarely used.

Gauss-Jordan Method

This method is the modification of Gauss elimination method. In this method, the variables are eliminated from both below and above the row to make the diagonal elements a_{11}, a_{22}, a_{33} unity and off-diagonal elements zero.

Illustrative Examples

Example 1: Use the Gauss elimination method to solve the system.

$$2x + 3y = 5$$

$$7x - y = 6$$

Solution

Given,

$$2x + 3y = 5 \quad \dots (i)$$

$$7x - y = 6 \quad \dots (ii)$$

Multiplying equation (i) by $\frac{7}{2}$ and subtracting from equation (ii), we have,

$$7x - y = 6$$

$$7x + \frac{21}{2}y = \frac{35}{2}$$

$$-\frac{23}{2}y = -\frac{23}{2} \quad \dots (iii)$$

Now, we have the following system of equations

$$2x + 3y = 5 \quad \dots (i)$$

$$-\frac{23}{2}y = -\frac{23}{2} \quad \dots (iii)$$

From equation (iii), we have $y = 1$.

Using the value of y in equation (i), we have,

$$2x + 3 \times 1 = 5$$

$$\text{or, } 2x = 2$$

$$\text{or, } x = 1$$

The required solution is $x = 1$ and $y = 1$.

Example 2. Solve by Gauss elimination method.

$$3x_1 + x_2 + x_3 = 5$$

$$x_1 - 4x_2 + x_3 = -2$$

$$x_1 + x_2 - 3x_3 = -1$$

Solution

Given equations are:

$$3x_1 + x_2 + x_3 = 5 \quad \dots(i)$$

$$x_1 - 4x_2 + x_3 = -2 \quad \dots(ii)$$

$$x_1 + x_2 - 3x_3 = -1 \quad \dots(iii)$$

Multiplying equation (i) by $\frac{1}{3}$ and subtracting from (ii)

$$x_1 - 4x_2 + x_3 = -2$$

$$x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 = \frac{5}{3}$$

$$\hline$$

$$-\frac{13}{3}x_2 - \frac{2}{3}x_3 = -\frac{11}{3} \quad \dots(iv)$$

Again, multiplying equation (i) by $\frac{1}{3}$ and subtracting from (iii),

$$x_1 + x_2 - 3x_3 = -1$$

$$x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 = \frac{5}{3}$$

$$\hline$$

$$\frac{2}{3}x_2 - \frac{10}{3}x_3 = -\frac{8}{3} \quad \dots(v)$$

Again, multiplying equation (iv) by $\frac{2}{13}$ and adding with equation (v).

$$-\frac{2}{3}x_2 + \frac{4}{39}x_3 = -\frac{22}{39}$$

$$\frac{2}{3}x_2 - \frac{10}{3}x_3 = -\frac{8}{3}$$

$$\hline$$

$$-\frac{126}{39}x_3 = -\frac{126}{39} \quad \dots(vi)$$

Now, we have the following system of equations.

$$3x_1 + x_2 + x_3 = 5 \quad \dots(i)$$

$$-\frac{13}{3}x_2 + \frac{2}{3}x_3 = -\frac{11}{3} \quad \dots(iv)$$

$$-\frac{126}{39}x_3 = -\frac{126}{39} \quad \dots(vi)$$

From equation (vi), we have, $x_3 = 1$

Using $x_3 = 1$ in (iv)

$$-\frac{13}{3}x_2 = -\frac{11}{3} - \frac{2}{3}$$

$$x_2 = 1$$

Again, using $x_3 = 1$ and $x_2 = 1$ in (i), we get

$$3x_1 + 1 + 1 = 5$$

$$x_1 = 1$$

$$x_1 = 1, x_2 = 1, x_3 = 1.$$

Example 3. Use the Gauss elimination method to solve the system.

$$x_1 + 2x_2 + 3x_3 = 2$$

$$x_1 + x_2 - x_3 = 1$$

$$2x_1 + 3x_2 + 2x_3 = 3$$

Solution

Given equations are

$$x_1 + 2x_2 + 3x_3 = 2 \quad \dots(i)$$

$$x_1 + x_2 - x_3 = 1 \quad \dots(ii)$$

$$2x_1 + 3x_2 + 2x_3 = 3 \quad \dots(iii)$$

Subtracting equation (i) from equation (ii), we have,

$$x_1 + x_2 - x_3 = 1$$

$$x_1 + 2x_2 + 3x_3 = 2$$

$$\hline$$

$$-x_2 - 4x_3 = -1 \quad \dots(iv)$$

Again, multiplying equation (i) by 2 and subtracting from equation (iii), we have,

$$2x_1 + 3x_2 + 2x_3 = 3$$

$$2x_1 + 4x_2 + 6x_3 = 4$$

$$\hline$$

$$-x_2 - 4x_3 = -1 \quad \dots(v)$$

Subtracting equation (iv) from equation (v),

$$-x_2 - 4x_3 = -1$$

$$-x_2 - 4x_3 = -1$$

$$\hline$$

$$0 = 0 \quad \dots(vi)$$

Now, we have the following three equations

$$x_1 + 2x_2 + 3x_3 = 2 \quad \dots(i)$$

$$-x_2 - 4x_3 = -1 \quad \dots(iv)$$

$$0 \cdot x_3 = 0 \quad \dots(vi)$$

The equation (vi) is true for all values of x_3 . The variable x_3 is called the **free variable**. So, x_3 can take any value. Hence, we have infinitely many solutions for such system of equations.

Thus, if $x_3 = k$, then from equation (iv),

$$x_2 = 1 - 4k$$

Again, putting the values of x_2 and x_3 in (i), we get, $x_1 = 5k$.

Example 4. Examine the consistency of the system. Solve if possible.

$$x + 2y + 3z = 1$$

$$4x + 5y + 6z = 3$$

$$7x + 8y + 9z = 5$$

Solution

Given equations are

$$x + 2y + 3z = 1 \quad \dots (i)$$

$$4x + 5y + 6z = 3 \quad \dots (ii)$$

$$7x + 8y + 9z = 5 \quad \dots (iii)$$

Multiplying equation (i) by 4 and subtracting from (ii), we have,

$$4x + 5y + 6z = 3$$

$$4x + 8y + 12z = 4$$

$$\begin{array}{r} - \\ -3y - 6z = -1 \end{array} \quad \dots (iv)$$

Again, multiplying equation (i) by 7 and subtracting from (iii), we have,

$$7x + 8y + 9z = 5$$

$$7x + 14y + 21z = 7$$

$$\begin{array}{r} - \\ -6y - 12z = -2 \end{array} \quad \dots (v)$$

Multiplying equation (iv) by 2 and subtracting from equation (v), we have,

$$-6y - 12z = -1$$

$$-6y - 12z = -2$$

$$\begin{array}{r} + \\ 0 = 1 \end{array} \quad \dots (vi)$$

Now, we have the following system of equations,

$$x + 2y + 3z = 1 \quad \dots (i)$$

$$-3y - 6z = -1 \quad \dots (iv)$$

$$0z = 1 \quad \dots (vi)$$

Here, no value of z satisfies the equation (vi). So, the system has no solution. Hence the system of equations is inconsistent.

Example 5. Solve the system using Gauss elimination partial pivoting.

$$4x_1 + 2x_2 - 3x_3 = 4$$

$$x_1 - x_2 + x_3 = 0$$

$$2x_1 + 4x_2 + x_3 = 7$$

Solution

Given equations are

$$4x_1 + 2x_2 - 3x_3 = 4 \quad \dots (i)$$

$$x_1 - x_2 + x_3 = 0 \quad \dots (ii)$$

$$2x_1 + 4x_2 + x_3 = 7 \quad \dots (iii)$$

Multiplying equation (i) by $\frac{1}{4}$ and subtracting from (ii), we have,

$$x_1 - x_2 + x_3 = 0$$

$$x_1 + \frac{1}{2}x_2 - \frac{3}{4}x_3 = 1$$

$$\begin{array}{r} - \\ -\frac{3}{2}x_2 + \frac{7}{4}x_3 = -1 \end{array} \quad \dots (iv)$$

Again, multiplying equation (i) by $\frac{1}{2}$ and subtracting from (iii), we have,

$$2x_1 + 4x_2 + x_3 = 7$$

$$2x_1 + x_2 - \frac{3}{2}x_3 = 2$$

$$\begin{array}{r} - \\ 3x_2 + \frac{5}{2}x_3 = 5 \end{array} \quad \dots (v)$$

Since $|\frac{3}{2}| < |3|$, interchanging equation (iv) and equation (v), we have,

$$3x_2 + \frac{5}{2}x_3 = 5$$

$$-\frac{3}{2}x_2 + \frac{7}{4}x_3 = -1$$

Multiplying the first equation by $-\frac{1}{2}$ and subtracting from second equation, we have,

$$-\frac{3}{2}x_2 + \frac{7}{4}x_3 = -1$$

$$-\frac{3}{2}x_2 - \frac{5}{4}x_3 = -\frac{5}{2}$$

$$\begin{array}{r} + \\ 3x_3 = \frac{3}{2} \end{array} \quad \dots (vi)$$

Now, we have the following system of equations,

$$4x_1 + 2x_2 - 3x_3 = 4 \quad \dots (i)$$

$$-\frac{3}{2}x_2 + \frac{7}{4}x_3 = -1 \quad \dots (iv)$$

$$3x_3 = \frac{3}{2} \quad \dots (vi)$$

From (vi), we have $x_3 = \frac{1}{2}$

Using the value of x_3 in (ii), we have,

$$-\frac{3}{2}x_2 + \frac{7}{4} \times \frac{1}{2} = -1$$

$$\text{or, } -\frac{3}{2}x_2 = -1 - \frac{7}{8}$$

$$\text{or, } -\frac{3}{2}x_2 = -\frac{15}{8}$$

$$\text{or, } x_2 = \frac{15}{8} \times \frac{2}{3}$$

$$\therefore x_2 = \frac{5}{4}$$

Again, using the values of x_2 and x_3 in equation (i), we have,

$$4x_1 + 2 \times \frac{5}{4} - 3 \times \frac{1}{2} = 4$$

$$\text{or, } 4x_1 = 4 + \frac{3}{2} - \frac{5}{2}$$

$$\text{or, } 4x_1 = 3$$

$$\text{or, } x_1 = \frac{3}{4}$$

The required solution is $x_1 = \frac{3}{4}$, $x_2 = \frac{5}{4}$ and $x_3 = \frac{1}{2}$.

Exercise 6.3

1. Solve the following system of equations by Gauss elimination method.

a. $3x_1 - 2x_2 = 1$

$$x_1 + 5x_2 = 6$$

c. $5x_1 - 8x_2 - 28 = 0$

$$3x_1 - 7x_2 - 5 = 0$$

e. $3x_1 + x_2 + x_3 = 5$

$$x_1 - 4x_2 + x_3 = -2$$

$$x_1 + x_2 - 3x_3 = -1$$

b. $2x + 3y = 4$

$$3x + 2y = -4$$

d. $x - 2y + 3z = 2$

$$2x - 3y + z = 1$$

$$3x - y + 2z = 9$$

f. $2x_2 + 3x_3 = 7$

$$3x_1 - 2x_2 + 2x_3 = 1$$

$$2x_1 + 3x_2 - 3x_3 = 5$$

2. Test the consistency of the following system. Solve if possible.

a. $x - y - 2z = -1$

$$2x + y + z = 2$$

$$3x + 2y + 9z = 4$$

c. $x_1 + 2x_2 + 3x_3 = 1$

$$4x_1 + 5x_2 + 6x_3 = 3$$

$$7x_1 + 8x_2 + 9x_3 = 5$$

b. $x_1 + x_2 - x_3 = 1$

$$2x_1 + 3x_2 + 3x_3 = 3$$

$$x_1 - 3x_2 + 3x_3 = 2$$

d. $x_1 + 2x_2 + 5x_3 = 4$

$$-2x_1 + x_2 + 3x_3 = 12$$

$$6x_1 - 3x_2 - 9x_3 = 24$$

e. $x_2 - 4x_3 = 8$

$$2x_1 - 3x_2 + 2x_3 = 1$$

$$5x_1 - 8x_2 + 7x_3 = 1$$

Using the Gauss elimination with partial pivoting, solve.

3. $x_1 + x_2 + x_3 = 6$

$$2x_1 - x_2 = 4$$

$$4x_1 - 3x_2 - x_3 = 5$$

Very Short Answer Questions

- What type of solution does this system: $x - y = 1$ and $x - y = 5$ have?
- In Gauss elimination method, what is the coefficients of the variables of the equation a_{ij} where $i = j$ called?
- Write the two steps used in Gauss elimination method.

Answers

- a. $x_1 = 1, x_2 = 1$ b. $x = -4, y = 4$ c. $x_1 = \frac{156}{11}, x_2 = \frac{59}{11}$

d. $x = 3, y = 2, z = 1$ e. $x_1 = 1, x_2 = 1, x_3 = 1$ f. $x_1 = 1, x_2 = 2, x_3 = 1$
- a. Consistent, $x = \frac{7}{20}, y = \frac{5}{4}, z = \frac{1}{20}$ b. Consistent, $x_1 = \frac{5}{4}, x_2 = -\frac{1}{24}, x_3 = \frac{5}{24}$

c. Consistent, infinitely many solutions, infinitely many solutions

d. Inconsistent e. Inconsistent
- $x_1 = 3, x_2 = 2, x_3 = 1$

6.7.2 Gauss Seidel Method

An iterative method for solving the system of linear equations named after Carl Friedrich Gauss (1777-1855) and Philipp L. Seidel (1821-1896) is Gauss-Seidel method. This method gives the user to control the round of error. For faster convergence, the number of iterations can be reduced with close initial guess.

A simple way for initial guess is to assume that they are zeros.

Now, we consider the following system of equations.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

where a_{ij} and b_i are real constants.

We may write the above system as follows:

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}} \dots (i)$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}} \dots (ii)$$

$$\text{and } x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}} \dots (iii)$$

While writing (i), (ii) and (iii) we may rearrange the system so that the coefficients of leading diagonal become non-zero.

Let us make the initial guess $(x_1, x_2, x_3) = (0, 0, 0)$.

Putting $x_2 = 0$ and $x_3 = 0$ in (i), we have,

$$x_1 = \frac{b_1}{a_{11}} = p_1 \text{ (suppose)}$$

Again, putting $x_1 = p_1, x_3 = 0$ in (ii), we have,

$$x_2 = \frac{b_2 - a_{21}p_1}{a_{22}} = q_1 \text{ (suppose)}$$

And, putting $x_1 = p_1, x_2 = q_1$ in (iii), we have,

$$x_3 = \frac{b_3 - a_{31}p_1 - a_{32}q_1}{a_{33}} = r_1 \text{ (suppose)}$$

The initial and first iteration solution can be tabulated as follows:

Iteration	x_1	x_2	x_3
0	p_1	0	0
1	p_1	q_1	r_1

Thus, after first iteration, the approximate solution is $x_1 = p, x_2 = q_1$ and $x_3 = r_1$.

We continue the process until the desired level of accuracy is met.

The absolute relative approximate error is

$$|E_a|_i = \left| \frac{\text{New } x_i - \text{Old } x_i}{\text{New } x_i} \right| \times 100 \text{ is less than pre-assigned tolerance for all unknowns.}$$

Diagonally Dominant

If the absolute value of the main diagonal coefficient is greater than the sum of the absolute values of other coefficients in the same row then the system of equations is said to be diagonally dominant. Thus, a system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \text{ is diagonally dominant if}$$

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|.$$

Example 1. Test whether the system of equations $12x + 3y - 5z = 1$, $x + 5y + 3z = 28$ and $3x + 7y + 13z = 1$ is diagonally consistent?

Solution

Here,

$$|12| > |3| + |-5| = 8$$

$$|15| > |1| + |3| = 4$$

$$|13| > |3| + |7| = 10$$

Since the absolute value of the diagonal coefficient is greater than the sum of the absolute values of the other coefficients in that row, so the system is diagonally dominant.

If the system is diagonally dominant, then the convergence of the solution is assured. Otherwise it may not give the solution within the pre-assigned tolerance.

Well conditioned and ill-conditioned system

A system of equations is said to be **well-conditioned** if a small change in the coefficient of the variable in the system of equations shows a small deviation in the solution. But, if a small change in the coefficient of the variable in the system of equations shows a larger deviation in the solution of the system of equations then the system of equations is said to be **ill-conditioned**.

For example:

Consider the system of equations

$$x - y = 1$$

$$x - 1.001y = 0$$

Solving, $x = 1001, y = 1000$

Now, changing the coefficient of y in second equation to 0.999 then

$$x - y = 1$$

$$x - 0.999y = 0$$

Solving, $x = -999, y = -1000$

Here, small change in coefficient results the very large change in the solution. Such system is called ill conditioned system.

Geometrically, the ill-conditioned system represents the straight lines that are almost parallel. Even a slight shift on any line can make a great move of the intersection point in ill conditioned system.

Illustrative Examples

Example 1. Using Gauss Seidel method, solve the equations:

$$3x + 2y = -9$$

$$2x - 3y = -6$$

Solution

From the given equations we have,

$$x = \frac{-9-2y}{3}, y = \frac{2x+6}{3}$$

Initially, $x = 0, y = 0$

Iteration I:

$$x = \frac{-9-2 \times 0}{3} = -3$$

$$y = \frac{2 \times (-3) + 6}{3} = 0$$

Iteration II:

$$x = \frac{-9-2 \times 0}{3} = -3$$

$$y = \frac{2 \times (-3) + 6}{3} = 0$$

From 1st and 2nd iterations, the values of x and y are equal, so $x = -3, y = 0$.

Example 2. Solve the following equations using Gauss-Seidel method:

$$3x_1 + x_2 = 5$$

$$x_1 - 3x_2 = 5$$

Solution

$$x_1 = \frac{1}{3}(5 - x_2)$$

$$x_2 = \frac{1}{3}(x_1 - 5)$$

Initially, $x_1 = 0, x_2 = 0$

1st iteration:

$$x_1 = \frac{1}{3}(5 - 0) = 1.667$$

$$x_2 = \frac{1}{3}(1.667 - 5) = -1.111$$

2nd iteration:

$$x_1 = \frac{1}{3}(5 + 1.111) = 2.037$$

$$x_2 = \frac{1}{3}(2.037 - 5) = -0.988$$

3rd iteration:

$$x_1 = \frac{1}{3}(5 + 0.988) = 1.996$$

$$x_2 = \frac{1}{3}(1.996 - 5) = -1.001$$

4th iteration:

$$x_1 = \frac{1}{3}(5 + 1.001) = 2.000$$

$$x_2 = \frac{1}{3}(2.000 - 5) = -1.000$$

5th iteration:

$$x_1 = \frac{1}{3}(5 + 1) = 2$$

$$x_2 = \frac{1}{3}(2 - 5) = -1$$

From 4th and 5th iterations, the value of x_1 and x_2 are equal.

$$\therefore x_1 = 2, x_2 = -1.$$

Example 3. Use the Gauss-Seidel method to solve the systems:

$$4x - y + z = 8$$

$$2x + 5y + 2z = 3$$

$$x + 2y + 4z = 11$$

Solution

From the given equations, we have,

$$x = \frac{1}{4}(8 + y - z),$$

$$y = \frac{1}{5}(3 - 2x - 2z)$$

$$z = \frac{1}{4}(11 - x - 2y)$$

Initially, $x = 0, y = 0, z = 0$

1st iteration:

$$x = \frac{1}{4}(8 + 0 - 0) = 2$$

$$y = \frac{1}{5}(3 - 2 \times 2 - 2 \times 0) = -0.2$$

$$z = \frac{1}{4}(11 - 2 + 2 \times 0.2) = 2.35$$

2nd iteration:

$$x = \frac{1}{4}(8 - 0.2 - 2.35) = 1.363$$

$$y = \frac{1}{5}(3 - 2 \times 1.363 - 2 \times 2.35) = -0.885$$

$$z = \frac{1}{4}(11 - 1.363 + 2 \times 0.885) = 2.852$$

3rd iteration:

$$x = \frac{1}{4}(8 - 0.885 - 2.852) = 1.066$$

$$y = \frac{1}{5}(3 - 2 \times 1.066 - 2 \times 2.852) = -0.967$$

$$z = \frac{1}{4}(11 - 1.066 + 2 \times 0.967) = 2.967$$

4th iteration:

$$x = \frac{1}{4}(8 - 0.967 - 2.967) = 1.017$$

$$y = \frac{1}{5}(3 - 2 \times 1.017 - 2 \times 2.967) = -0.994$$

$$z = \frac{1}{4}(11 - 1.017 + 2 \times 0.994) = 2.993$$

5th iteration:

$$x = \frac{1}{4}(8 - 0.994 - 2.993) = 1.003$$

$$y = \frac{1}{5}(3 - 2 \times 1.003 - 2 \times 2.993) = -0.998$$

$$z = \frac{1}{4}(11 - 1.003 + 2 \times 0.998) = 2.998$$

6th iteration:

$$x = \frac{1}{4}(8 - 0.998 - 2.998) = 1.001$$

$$y = \frac{1}{5}(3 - 2 \times 1.001 - 2 \times 2.998) = -1.000$$

$$z = \frac{1}{4}(11 - 1.001 + 2 \times 1.000) = 3.000$$

7th iteration:

$$x = \frac{1}{4}(8 - 1 - 3) = 1$$

$$y = \frac{1}{5}(3 - 2 \times 1 - 2 \times 3) = -1$$

$$z = \frac{1}{4}(11 - 1 + 2 \times 1) = 3$$

From 6th and 7th iterations, the values of x , y and z are equal, so $x = 1$, $y = -1$ and $z = 3$.

Exercise 6.4

Examine whether the following system of equations are diagonally dominant?

a. $\begin{cases} 3x_1 + x_2 + x_3 = 5 \\ -x_1 + 5x_2 + 2x_3 = 6 \\ -2x_1 - x_2 + 5x_3 = 2 \end{cases}$

b. $\begin{cases} x + 3y - z = 5 \\ 2x - 5y + 2z = 1 \\ 3x + 2y + 7z = 1 \end{cases}$

Solve the given system of equations by Gauss-Seidel method in second iteration.

a. $\begin{cases} 3x_1 + x_2 = 4 \\ x_1 - 2x_2 = 3 \end{cases}$

b. $\begin{cases} -4x_1 + x_2 - x_3 = -8 \\ 3x_1 + 6x_2 + 2x_3 = 1 \\ x_1 - x_2 + 3x_3 = 2 \end{cases}$

Solve the following system of equations using Gauss-Seidel method.

a. $\begin{cases} 3x_1 - 2x_2 = 1 \\ -x_1 + 4x_2 = 3 \end{cases}$

b. $\begin{cases} 3x_1 + x_2 = 5 \\ x_1 + 2x_2 = 5 \end{cases}$

c. $\begin{cases} 3x + y - z = 2 \\ 2x - 5y + z = 20 \\ x - 3y - 8z = 3 \end{cases}$

d. $\begin{cases} 8x_1 + x_2 + x_3 = 13 \\ 2x_1 + 5x_2 + x_3 = 15 \\ -x_1 + 4x_2 - 7x_3 = -14 \end{cases}$

Very Short Answer Questions

- Define ill conditioned system of linear equations.
- Define well conditioned system of linear equations.
- When a system of equations is said to be diagonally dominant?
- What is a simple way for initial guess in Gauss Seidel method?
- Find the solution of the system of equation $3x_1 + x_2 = 5$ and $x_1 + 2x_2 = 5$ in second iteration using Gauss Seidel method.

Answers

- | | |
|-----------------------------------|--|
| 1. a. Diagonally dominant | b. Not diagonally dominant |
| 2. a. $x_1 = 1.611, x_2 = -0.694$ | b. $x_1 = 1.861, x_2 = -0.671, x_3 = -0.177$ |
| 3. a. $x_1 = 1, x_2 = 1$ | b. $x_1 = 1, x_2 = 2$ |
| d. $x_1 = 1, x_2 = 2, x_3 = 3$ | c. $x = 2, y = -3, z = 1$ |

Very Short Answer Questions

- 1.111, 1.945

6.7.3 Matrix Inversion Method

We recall about the inverse of a square matrix which has been studied in first semester. A square matrix A is **singular** if $|A| = 0$ and **non-singular** if $|A| \neq 0$. The inverse of a non-singular square matrix A , denoted by A^{-1} , is such that

$$AA^{-1} = A^{-1}A = I \text{ where } I \text{ is the identity matrix.}$$

Let us consider the following system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

In matrix notation, the above system can be written as

$$AX = B \quad \dots (i)$$

$$\text{where, } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If $|A| \neq 0$, then (i) can be written as

$$A^{-1}(AX) = A^{-1}B$$

$$\text{or, } (A^{-1}A)X = A^{-1}B$$

$$\text{or, } IX = A^{-1}B$$

$$\therefore X = A^{-1}B.$$

This equation gives the values of x_1 , x_2 and x_3 . The A^{-1} can be calculated using Gauss-Jordan method with the following steps.

Step 1: Augment the coefficient matrix A with an identity matrix

$$\text{i.e. } \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & : & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & : & 0 & 0 & 1 \end{bmatrix}$$

Step 2: Apply Gauss-Jordan method to the augmented matrix to reduce A to an identity matrix I as

$$\begin{bmatrix} 1 & 0 & 0 & : & a'_{11} & a'_{12} & a'_{13} \\ 0 & 1 & 0 & : & a'_{21} & a'_{22} & a'_{23} \\ 0 & 0 & 1 & : & a'_{31} & a'_{32} & a'_{33} \end{bmatrix}$$

Step 3: The right hand side of matrix in step 2 is A^{-1} .

$$\text{Thus, } A^{-1} = \begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{bmatrix}$$

Illustrative Examples

Example 1. Apply Gauss-Jordan method to find the inverse of $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.

Solution

We augment A with identity matrix I

$$\text{i.e. } [A : I] = \begin{bmatrix} 2 & 5 & : & 1 & 0 \\ 1 & 3 & : & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & : & 0 & 1 \\ 2 & 5 & : & 1 & 0 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 3 & : & 0 & 1 \\ 0 & -1 & : & 1 & -2 \end{bmatrix} R_2 \rightarrow R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 3 & : & 0 & 1 \\ 0 & 1 & : & -1 & 2 \end{bmatrix} R_2 \rightarrow (-1)R_2$$

$$\sim \begin{bmatrix} 1 & 0 & : & 3 & -5 \\ 0 & 1 & : & -1 & 2 \end{bmatrix} R_1 \rightarrow R_1 - 3R_2$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Example 2. Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by Gauss-Jordan Method.

Solution

We augment A with identity matrix I .

$$[A : I] = \begin{bmatrix} 2 & 1 & 1 & : & 1 & 0 & 0 \\ 3 & 2 & 1 & : & 0 & 1 & 0 \\ 2 & 1 & 2 & : & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & : & \frac{1}{2} & 0 & 0 \\ 3 & 2 & 1 & : & 0 & 1 & 0 \\ 2 & 1 & 2 & : & 0 & 0 & 1 \end{bmatrix} R_1 \rightarrow \frac{1}{2}R_1$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & : & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & : & -1 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & : & -3 & 2 & 0 \\ 0 & 0 & 1 & : & -1 & 0 & 1 \end{bmatrix} R_2 \rightarrow 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & : & 2 & -1 & 0 \\ 0 & 1 & -1 & : & -3 & 2 & 0 \\ 0 & 0 & 1 & : & -1 & 0 & 1 \end{bmatrix} R_1 \rightarrow R_1 - \frac{1}{2}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 3 & -1 & -1 \\ 0 & 1 & 0 & : & -4 & 2 & 1 \\ 0 & 0 & 1 & : & -1 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 + R_3 \\ R_1 \rightarrow R_1 - R_3 \end{matrix}$$

Thus, $A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

Example 3. Solve the following system of equations by inverse matrix method.

$$3x - y = 5$$

$$8x - 3y = 14.$$

Solution

The matrix form of given system is

$$\begin{bmatrix} 3 & -1 \\ 8 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 14 \end{bmatrix}$$

$$\text{i.e. } AX = B$$

$$X = A^{-1}B \quad \dots (i)$$

$$\text{where, } A = \begin{bmatrix} 3 & -1 \\ 8 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 14 \end{bmatrix}$$

We augment A with identity matrix I.

$$[A|I] = \begin{bmatrix} 3 & -1 & : & 1 & 0 \\ 8 & -3 & : & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -\frac{1}{3} & : & \frac{1}{3} & 0 \\ 8 & -3 & : & 0 & 1 \end{bmatrix} R_1 \rightarrow \frac{1}{3}R_1$$

$$\sim \begin{bmatrix} 1 & -\frac{1}{3} & : & \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & : & -\frac{8}{3} & 1 \end{bmatrix} R_2 \rightarrow R_2 - 8R_1$$

$$\sim \begin{bmatrix} 1 & -\frac{1}{3} & : & \frac{1}{3} & 0 \\ 0 & 1 & : & 8 & -3 \end{bmatrix} R_2 \rightarrow (-3)R_2$$

$$\sim \begin{bmatrix} 1 & 0 & : & 3 & -1 \\ 0 & 1 & : & 8 & -3 \end{bmatrix} R_1 \rightarrow R_2 + \frac{1}{3}R_2$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & -1 \\ 8 & -3 \end{bmatrix}$$

From (i),

$$X = \begin{bmatrix} 3 & -1 \\ 8 & -3 \end{bmatrix} \begin{bmatrix} 5 \\ 14 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15-14 \\ 40-42 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$x = 1, y = -2.$$

Example 4. Solve the following system of equations using inverse matrix method.

$$-2x + 2y + z = -4$$

$$-8x + 7y - 4z = -47$$

$$9x - 8y + 5z = 55.$$

Solution

The matrix form of the given system is

$$\begin{bmatrix} -2 & 2 & 1 \\ -8 & 7 & -4 \\ 9 & -8 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ -47 \\ 55 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B \quad \dots (i)$$

We augment A with the identity matrix I.

$$[A|I] = \begin{bmatrix} -2 & 2 & 1 & : & 1 & 0 & 0 \\ -8 & 7 & -4 & : & 0 & 1 & 0 \\ 9 & -8 & 5 & : & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -\frac{1}{2} & : & -\frac{1}{2} & 0 & 0 \\ -8 & 7 & -4 & : & 0 & 1 & 0 \\ 9 & -8 & 5 & : & 0 & 0 & 1 \end{bmatrix} R_1 \rightarrow \left(-\frac{1}{2}\right)R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -\frac{1}{2} & : & -\frac{1}{2} & 0 & 0 \\ 0 & -1 & -8 & : & -4 & 1 & 0 \\ 0 & 1 & \frac{19}{2} & : & \frac{9}{2} & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 + 8R_1 \\ R_3 \rightarrow R_3 - 9R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -\frac{1}{2} & : & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 8 & : & 4 & -1 & 0 \\ 0 & 1 & \frac{19}{2} & : & \frac{9}{2} & 0 & 1 \end{bmatrix} R_2 \rightarrow (-1)R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{15}{2} & : & \frac{7}{2} & -1 & 0 \\ 0 & 1 & 8 & : & 4 & -1 & 0 \\ 0 & 1 & \frac{3}{2} & : & \frac{1}{2} & 1 & 1 \end{bmatrix} \begin{matrix} R_1 \rightarrow R_1 + R_2 \\ R_3 \rightarrow R_3 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{15}{2} & : & \frac{7}{2} & -1 & 0 \\ 0 & 1 & 8 & : & 4 & -1 & 0 \\ 0 & 0 & 1 & : & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} R_3 \rightarrow \frac{2}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -5 \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{19}{3} & -\frac{16}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{array} \right] \begin{array}{l} R_1 \rightarrow R_1 - \frac{15}{2}R_3 \\ R_2 \rightarrow R_2 - 8R_3 \end{array}$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -6 & -5 \\ \frac{4}{3} & -\frac{19}{3} & -\frac{16}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

Now, from (i)

$$X = A^{-1}B$$

$$\text{or, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -6 & -5 \\ \frac{4}{3} & -\frac{19}{3} & -\frac{16}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} -4 \\ -47 \\ 55 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 + 6 \times 47 - 5 \times 55 \\ \frac{4}{3} \times (-4) + \left(-\frac{19}{3}\right) \times (-47) + \left(-\frac{16}{3}\right) \times 55 \\ \frac{1}{3} \times (-4) + \frac{2}{3} \times (-47) + \frac{2}{3} \times 55 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

$$\therefore x=3, y=-1, z=4.$$

Exercise 6.5

1. Find the inverses of following matrices using Gauss-Jordan method.

a. $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$

b. $\begin{pmatrix} 3 & 6 \\ 7 & 2 \end{pmatrix}$

c. $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

2. Solve the following system of equations by inverse matrix method.

a. $x + 2y = 4$

$$3x - y = 5$$

b. $2x + 5y = 3$

$$x + 3y = 2$$

c. $2x_1 + 3x_2 + 4 = 0$

$$3x_1 = 5 + 2x_2$$

d. $x_1 - 2x_2 - x_3 = 1$

$$x_1 - x_2 + 2x_3 = 9$$

$$2x_1 - 3x_2 - x_3 = 4$$

e. $x + y + z = 9$

$$2x + 5y + 7z = 52$$

$$2x + y - z = 0$$

$$2x + 3y = 4$$

$$4x - z = 5$$

$$4y + 3z = -5$$

f. $9y - 5x = 3$

$$x + z = 1$$

$$z + 2y = 2$$

Very Short Answer Questions

Define singular matrix.

If $A = \begin{bmatrix} 2 & 5 \\ a & 3 \end{bmatrix}$ does not have an inverse then find a .

Find the inverse of $\begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$.

Answers

1. a. $\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$ b. $\begin{pmatrix} \frac{1}{18} & \frac{1}{6} \\ \frac{7}{36} & -\frac{1}{12} \end{pmatrix}$ c. $\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

2. a. $x=2, y=1$ b. $x=-1, y=1$ c. $x=\frac{7}{13}, y=-\frac{12}{13}$

d. $x_1=2, x_2=-1, x_3=3$

e. $x=1, y=3, z=5$

f. $x=3, y=2, z=-2$ g. $x=\frac{1}{2}, y=1, z=-3$

Very Short Answer Questions

2. $\frac{6}{5}$

3. $\begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$

6.8 Bisection Method**Computational Methods**

Numerical computational method plays an important role in solving many real life problem mathematically. Numerical computing is an approach for solving complex mathematical problems using simple arithmetic operations. An analytic solutions are often either too cumbersome or do not exist, we have to find an approximate method of solutions. Numerical methods give approximate solution to the problems. They are much easier when compared to the analytical (exact) methods. Any numerical computation is judged on the basis of accuracy, rate of convergence, numerical stability and efficiency.

1. **Accuracy:** Accuracy refers to the degree to which a quantity is correct and free from the error. There are two kinds of numbers (i) exact numbers and (ii) approximate numbers. The numbers are $1, 3, \frac{2}{5}, \sqrt{4}$ etc. are exact numbers. Approximate numbers are those numbers that represent the numbers to a certain degree of accuracy. An approximate value of π is 3.14159 or better approximate value is 3.14159296. We can not write the exact value of π . When come across numbers with a large number of digits, we cut them to a usable number of figures. So, numerical computing is judged on the basis of accuracy.
2. **Rate of convergence:** Most of the numerical computing are iterative in nature. Rate of convergence refers to the speed at which a convergent sequence approaches its limit.

Let us suppose that a sequence $\{x_k\}$ converges to L .

The sequence $\{x_k\}$ is said to converge linearly to L if there exists a number $N \in (0, 1)$ such that

$$\lim_{k \rightarrow \infty} \frac{|x_{k+1} - L|}{|x_k - L|} = N$$

The number N is called the rate of convergence.

If the sequence converges and $N = 0$, the sequence is said to converge super-linearly and $N = 1$, the sequence is said to converge sub-linearly.

3. **Numerical stability:** An algorithm is said to be numerically stable if an error does not grow to much larger during the computation. This happens when the problem is well-conditioned.
4. **Efficiency:** Efficiency refers to the number of iterations in the algorithm, amount of effort and time. A method that requires less time and less effort to get the desired accuracy is preferable method.

Approximations and Error in Numerical Computing

In numerical computing, the error may arise differently. The major sources of errors are:

1. **Initial errors:** If the initial data is not correct then the deviation from their respective exact values are called initial error.
2. **Truncation errors:** Truncation error refers to the error in a method. Sometimes, a numerical computation may involve infinite process. Since infinite process cannot be carried out indefinitely, one is forced to stop the process and hence accept an approximate solution. The error caused by this process is truncation error.

Rounding off error: In numerical computation, we come across the numbers which have large number of digits and it is necessary cut after certain place to a usable number of figures. The errors occurred by this process is called rounding off error. It occurs due to the inability of calculating devices to perform exact arithmetic.

Measurement of Errors

The errors can be measured in the following ways:

True error: The difference between true (actual) value and near (approximate) value is called true error. Thus,

$$\text{True error} = \text{Exact value} - \text{Approximate value}$$

Absolute error: The absolute error is defined by

$$|\text{Exact value} - \text{Approximate value}|$$

Relative error: The relative error is given by the formula

$$\left| \frac{\text{Exact value} - \text{Approximate value}}{\text{Exact value}} \right| \text{ provided that exact value is not equal to zero.}$$

Percent error: The percent error is given by "Relative Error $\times 100$ "

$$\text{i.e. } \left| \frac{\text{Exact value} - \text{Approximate value}}{\text{Exact value}} \right| \times 100 \text{ provided that exact value is not zero.}$$

6.8.1 Non-linear Equations

A linear equation is an equation of the form $y = ax + b$, $a \neq 0$ where a and b are constants. An equation which is not linear is called **non-linear equation**. For example: $2x^2 + 7xy + 5y^2 = 0$, $x^2 + 5x + 6 = 0$ are non-linear equations. These are algebraic equations. The equations $\sin x = 0$, $e^x = 1$, $\cos x = x$ are transcendental.

Bisection Method

In mathematics, it is a root finding algorithm which bisects an interval repeatedly and select a sub-interval in which a root lie for the further processing. It is very simple method of numerical computation but slow. The method is based on the following theorems.

Intermediate Value Theorem

Let f be continuous on the interval $[a, b]$. If $f(a) \neq f(b)$, then for every k between $f(a)$ and $f(b)$, there is a point $c \in (a, b)$ such that $f(c) = k$.

Bolzano Theorem

Let $f(x)$ be a continuous function on $[a, b]$. If $f(a)f(b) < 0$ then an equation $f(x)$ has at least one root between a and b .

Bisection Method Procedure

Let f be a continuous function defined on $[a, b]$ with $f(a) \cdot f(b) < 0$. By Bolzano's theorem, there exists a root in (a, b) . We assume that $f(x)$ has only one root in (a, b) . Bisect the interval $[a, b]$ to get a mid-point $x_0 = \frac{a+b}{2}$. If $f(x_0) = 0$ then x_0 is a root, which is very unlikely but possible sometimes.

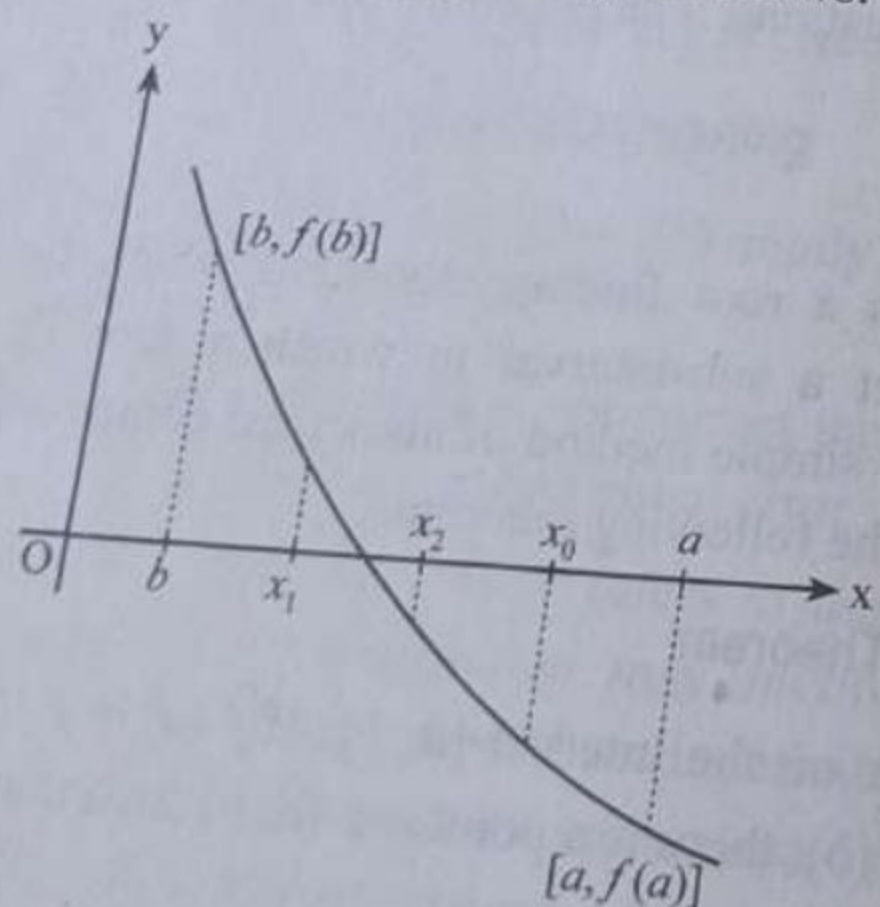
If $f(x_0) \neq 0$ then the root lies either between a and x_0 or between x_0 and b depending on whether $f(x_0)$ is positive or negative. We denote the new interval by $[a_1, b_1]$ whose length is $\frac{|b-a|}{2}$.

Next, we bisect $[a_1, b_1]$, say, as $x_1 = \frac{a_1+b_1}{2}$. If $f(x_1) = 0$ then x_1 is the required root. If $f(x_1) \neq 0$ then we repeat the above process. The new interval containing the root has the length $\frac{|b-a|}{2^2}$.

We repeat the process until the desired interval is found, the length of the interval is less than, say ϵ .

Clearly, the interval width is reduced by a factor of one-half at each step. At the end of n^{th} step, the new interval will be $[a_n, b_n]$ of length $\frac{|b-a|}{2^n}$. Thus we have $\frac{|b-a|}{2^n} > \epsilon$.

The bisection method is shown graphically as follows:



We stop the process of bisecting the interval $[a_k, b_k]$ in one of the following three cases.

1. Maximum number of iterations allowed
2. When $f(x_k) = 0$
3. When $\frac{|b_k - a_k|}{2} < \epsilon$, where $\epsilon > 0$ is the pre-assigned error tolerance.

Illustrative Examples

Example 1. Apply the method of successive bisection to find the square roots of 5 within 2 places of decimal in $[2, 3]$.

Solution

Let x be the square roots of 5. Then,

$$x^2 = 5$$

$$x^2 - 5 = 0$$

or, Let $f(x) = x^2 - 5$. Given, $a = 2$ and $b = 3$.

$$\text{Here, } f(2) = 2^2 - 5 = -1$$

$$f(3) = 3^2 - 5 = 4$$

Since $f(2)$ and $f(3)$ have opposite sign, so a root lies between 2 and 3.

a	b	$m = \frac{a+b}{2}$	$f(a)$	$f(b)$	$f(m)$
2	3	2.5	-1	4	1.25
2	2.5	2.25	-1	1.25	0.0625
2	2.25	2.125	-1	0.0625	-0.4844
2.125	2.25	2.1875	-0.4844	0.0625	-0.2148
2.1875	2.25	2.2188	-0.2148	0.0625	-0.0769
2.2188	2.25	2.2344	-0.0769	0.0625	-0.0075
2.2344	2.25	2.2422	-0.0075	0.0625	0.0275
2.2344	2.2422	2.2383	-0.0075	0.0275	0.0099
2.2344	2.2383		-0.0075	0.0099	

Since a and b have same value upto 2 places of decimal, so the required root is 2.23.

Example 2. Using method of bisection, find the root of the equation $x^3 - x - 4 = 0$ lying between 1 and 2 correct to 3 places of decimals.

Solution

Given equation is $x^3 - x - 4 = 0$

Let

$$f(x) = x^3 - x - 4$$

$$f(1) = 1^3 - 1 - 4 = -4$$

$$f(2) = 2^3 - 2 - 4 = 2$$

Since $f(1)$ and $f(2)$ have opposite signs, a root lies between 1 and 2.

a	b	$m = \frac{a+b}{2}$	$f(a)$	$f(b)$	$f(m)$
1	2	1.5	-4	2	-2.125
1.5	2	1.75	-2.125	2	-0.3906
1.75	2	1.875	-0.3906	2	0.7167
1.75	1.875	1.8125	-0.3906	0.7167	0.1418
1.75	1.8125	1.7812	-0.3906	0.1418	-0.1296
1.7812	1.8125	1.7969	-0.1296	0.1418	0.0048
1.7812	1.7969	1.7890	-0.1296	0.0048	-0.0625
1.7890	1.7969	1.7929	-0.0625	0.0048	-0.0292
1.7929	1.7969	1.7949	-0.0292	0.0048	-0.0121
1.7949	1.7969	1.7959	-0.0121	0.0048	-0.0036
1.7959	1.7969	1.7964	-0.0036	0.0048	0.006
1.7959	1.7964	1.7961	-0.0036	0.0006	-0.0014
1.7961	1.7964		-0.0014	0.0006	

Since a and b have same value upto 3 places of decimal, so the required root is 1.796.

Example 3. Use the Bisection method to find the solution of the equation $x - 2^{-x} = 0$ in the interval $[0, 1]$, accurate to within 10^{-3} .

Solution

Let

$$f(x) = x - 2^{-x} = x - \frac{1}{2^x}$$

$$f(0) = 0 - \frac{1}{2^0} = -1$$

$$f(1) = 1 - \frac{1}{2} = 0.5$$

Since $f(0) \times f(1) = (-1) \times 0.5 = -0.5 < 0$, a root lies between 0 and 1.

a	b	$m = \frac{a+b}{2}$	$f(a)$	$f(b)$	$f(m)$
0	1	0.5	-1	0.5	-0.2071
0.5	1	0.75	-0.2071	0.5	0.1554
0.5	0.75	0.625	-0.2071	0.1554	-0.0234
0.625	0.75	0.6875	-0.0234	0.1554	0.06657
0.625	0.6875	0.65625	-0.0234	0.06657	0.02172
0.625	0.65625	0.640625	-0.0234	0.02172	-0.00081

Here, $|f(m)| = |-0.00081| = 0.00081 < 10^{-3} = 0.001$

So, required root is 0.640625.

Example 4. How many iterations are required to get the root with the tolerance 10^{-3} in the interval $[1, 2]$?

Solution

Here, $a = 1, b = 2$

Error tolerance (ϵ_i) = 10^{-3}

We have,

$$\frac{|b-a|}{2^i} < \epsilon_i$$

$$\frac{2-1}{2^i} < 10^{-3}$$

$$2^{-i} < 10^{-3}$$

$$-i \log_{10} 2 < -3 \log_{10} 10$$

$$i > \frac{3}{\log_{10} 2} = 9.96578$$

Hence, required number of iterations is 10.

Exercise 6.6

- Apply the method of bisection to find the
 - square root of 2 within two places of decimal in $[1, 2]$.
 - the approximate value of $\sqrt[3]{7}$ within an error of 10^{-3} .
 - cube root of 50 within 2 places of decimal in $[3, 4]$.
- Using the bisection method to find the root of the equation.
 - $x^2 + x - 4 = 0$ in $[1, 2]$ correct to 2 places of decimals.
 - $x^3 - 4x + 1 = 0$ in $[1, 2]$ correct to 2 places of decimals.
 - $x^3 - 2x - 5 = 0$ lying between 2 and 3 correct to 3 places of decimals with error less than 0.001.
 - $2x^3 - 5x + 2 = 0$ lying between 1 and 2 with error less than 10^{-2} .
 - $\cos x = 3x - 1$ correct to 4 places of decimals.
 - $e^{-x} = 10x$ correct to 4 places of decimals.
- Show that the equation $f(x) = x^3 - 10 = 0$ has no negative root and one positive root. Find the positive root using bisection method with $a = 2, b = 3$ and $\epsilon = 0.01$.
- How many iterations are required to get the root with the tolerance 10^{-4} in the interval $[1, 2]$?

Very Short Answer Questions

- Write the condition for bisection method for the function $f(x)$ on the interval $[a, b]$.
- What do you mean by non-linear equation?
- Why we can't use bisection method to the function $f(x) = x^2$ though it has a root $x = 0$?

Answers

- | | | | |
|-------------|-----------|-----------|------------|
| 1. a. 1.41 | b. 1.91 | c. 3.68 | |
| 2. a. 1.56 | b. 1.86 | c. 2.094 | d. 1.32032 |
| | e. 0.6071 | f. 0.0913 | |
| 3. 2.154297 | | | |
| 4. 14 | | | |

6.9 Newton-Raphson Method

Newton-Raphson method is another iterative method for solving the equation $f(x) = 0$, where f is assumed to have a continuous derivative f' .

This method is commonly used because of its great speed and simplicity. This method depends on an initial guess which is reasonably close to the exact root. Let the initial guess be x_0 .

Draw a tangent to the curve at $x = x_0$. The point of intersection of this tangent with x -axis gives the second approximation to the root. Let x_1 be the point of intersection of the tangent and the x -axis. The slope of the tangent at $(x_0, f(x_0))$ is given by

$$\tan \theta = \frac{f(x_0) - 0}{x_0 - x_1} = f'(x_0)$$

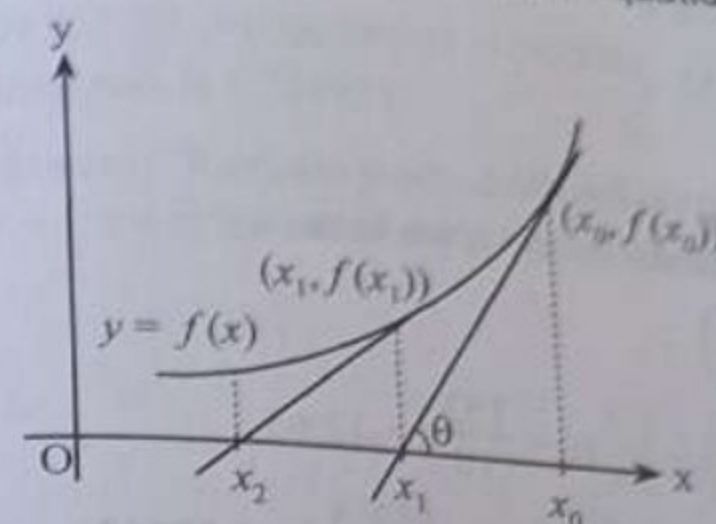
$$\text{or, } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The next approximation will be

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



We stop the process of finding the roots within an error tolerance $\varepsilon > 0$ if one of the following condition is met

- $|x_n - x_{n-1}| < \varepsilon$
- $|f(x_n)| < \varepsilon$
- $\frac{|x_n - x_{n-1}|}{|x_n|} < \varepsilon$

Finding square roots by Newton-Raphson method

Let a be the square root of x .

$$\text{Then, } x^2 - a = 0$$

$$\text{Let } f(x) = x^2 - a$$

$$f'(x) = 2x.$$

We have,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^2 - a}{2x_n} \\ &= \frac{x_n^2 + a}{2x_n} \\ &= \frac{1}{2} \left(x_n + \frac{a}{x_n} \right). \end{aligned}$$

Illustrative Examples

Example 1. Using Newton-Raphson's method, find the square root of 153 correct to 3 places of decimals.

Solution

Let x be the square root of 153.

$$\text{Then, } x^2 = 153$$

$$x^2 - 153 = 0$$

$$\text{Let } f(x) = x^2 - 153$$

$$\therefore a = 153$$

Take, initial guess $(x_0) = 10$

By Newton-Raphson's method, we have

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right)$$

$$\therefore x_1 = \frac{1}{2} \left(x_0 + \frac{a}{x_0} \right) = \frac{1}{2} \left(10 + \frac{153}{10} \right) = 12.65$$

$$\therefore x_2 = \frac{1}{2} \left(x_1 + \frac{a}{x_1} \right) = \frac{1}{2} \left(12.65 + \frac{153}{12.65} \right) = 12.37243$$

$$\therefore x_3 = \frac{1}{2} \left(x_2 + \frac{a}{x_2} \right) = \frac{1}{2} \left(12.37243 + \frac{153}{12.37243} \right) = 12.36931$$

$$\therefore x_4 = \frac{1}{2} \left(x_3 + \frac{a}{x_3} \right) = \frac{1}{2} \left(12.36931 + \frac{153}{12.36931} \right) = 12.36931$$

Comparing the values of x_3 and x_4 , we find that the digits in the first three places of decimal are same. Hence, the required square roots of 153 to three places of decimal is 12.369.

Example 2. Use Newton-Raphson's method to approximate $\sqrt[3]{2}$ with an error less than 0.00001.

Solution

$$\text{Let, } x = \sqrt[3]{2}$$

$$\text{or, } x^3 = 2$$

$$\text{or, } x^3 - 2 = 0$$

$$\text{Let } f(x) = x^3 - 2$$

$$f'(x) = 3x^2$$

We have,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let us take an initial guess $(x_0) = 1$.

Now,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1^3 - 2}{3 \cdot 1^2} = 1.333333$$

$$|f(x_1)| = |(1.333333)^3 - 2| = 0.370369 > 0.00001.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.333333 - \frac{(1.333333)^3 - 2}{3 \cdot (1.333333)^2} = 1.263889$$

$$|f(x_2)| = |(1.263889)^3 - 2| = 0.018955 > 0.00001$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.263889 - \frac{(1.263889)^3 - 2}{3 \cdot (1.263889)^2} = 1.259933$$

$$|f(x_3)| = |(1.259933)^3 - 2| = 0.000059 > 0.00001$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.259933 - \frac{(1.259933)^3 - 2}{3 \cdot (1.259933)^2} = 1.259921$$

Access Specimen

$$|f(x_4)| = |(1.259921)^3 - 2| = 0.0000001 < 0.00001$$

Hence the required root is 1.259921

Example 3. Use Newton-Raphson method to find the solution of the equation $x^3 + x - 1 = 0$ in the interval $[0, 1]$, accurate to within 10^{-4} .

Solution

Here,

$$f(x) = x^3 + x - 1$$

$$f'(x) = 3x^2 + 1$$

$$f(0) = -1$$

$$f(1) = 1$$

Since $f(0)$ and $f(1)$ have opposite signs, a real root lies between 0 and 1. Let us take initial guess $x_0 = 1$.

By Newton-Raphson's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1}{4} = 0.75$$

$$|f(x_1)| = 0.17875 > 10^{-4}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.75 - \frac{0.171875}{2.6875} = 0.68605$$

$$|f(x_2)| = 0.00895 > 10^{-4}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.68605 - \frac{0.00895}{2.41199} = 0.68234$$

$$|f(x_3)| = 0.00003 < 10^{-4}$$

Hence, the required root is 0.68234.

Exercise 6.7

1. If $f(x) = x^2 - 2$ and $x_0 = 1$, find x_1 , x_2 and x_3 using Newton-Raphson method.
2. Find (a) square root of 51 (b) cube roots of 123 correct to 3 places of decimals using Newton-Raphson method.
3. Using Newton-Raphson method, find a root of
 - a. $x^3 - x - 4 = 0$ between 1 and 2 to three places of decimal.
 - b. $x^3 - 2x - 5 = 0$ lying between 2 and 3 correct to 3 places of decimals.
 - c. $2x^2 - 3x - 1 = 0$ taking $x_0 = 1$ with error less than 10^{-4} .
 - d. $x - \cos x = 0$ to 3 places of decimals.
 - e. $x + \log x = 20$ to 3 places of decimals.

Very Short Answer Questions

1. Write the general formula for Newton-Raphson method.
2. If a be the square root of x then write the Newton-Raphson's formula for square root.
3. Find the next iterative value of the root of $x^2 - 4 = 0$ by Newton-Raphson method if the initial guess is 1.

Answers

1. $x_1 = 1.5, x_2 = 1.4166, x_3 = 1.4142$
2. a. 7.141 b. 4.973
3. a. 1.796 b. 2.094 c. 1.7808 d. 0.739
e. 1.756

Very Short Answer Questions

3. 2.5

Appendix

Mathematical Software (Mathematica, Python and MATLAB)

The most popular softwares used in mathematics are MATLAB, Mathematica, Python and Maple. The websites for these software systems are MATLAB (www.mathworks.com), Mathematica (www.wolfram.com), Maple (www.maplesoft.com) and Python (www.python.org). There are many other mathematical software systems. But each of these systems has its own syntax. As per the demand of course, Mathematica, MATLAB and Python are introduced.

According to Wikipedia, the free encyclopedia, "Wolfram Mathematica (usually termed Mathematica) is a modern technical computing system spanning most areas of technical computing including neural networks, machine learning, image processing, geometry, data science, visualizations, and others. The system is used in many technical, scientific, engineering, mathematical, and computing fields. It was conceived by Stephen Wolfram and is developed by Wolfram Research of Champaign, Illinois. The Wolfram Language is the programming language used in Mathematica."

According to Wikipedia, the free encyclopedia "MATLAB (matrix laboratory) is a multi-paradigm numerical computing environment and proprietary programming language developed by Mathworks. MATLAB allows matrix manipulations, plotting of functions and data, implementations of algorithms, creation of user interfaces, and interfacing with programs written in other languages, including C, C++, C#, Java, Fortran and Python.

Although MATLAB is intended primarily for numerical computing, an optional toolbox uses the MuPAD symbolic engine, allowing access to symbolic computing abilities. An additional package, Simulink, adds graphical multi-domain simulation and model-based design for dynamic and embedded systems.

MATLAB has more than 3 millions users worldwide. MATLAB users come from various backgrounds of engineering, science, and economics."

Questions

Newton-Raphson method.

When write the Newton-Raphson's formula

of the root of $x^2 - 4 = 0$ by Newton-
ess is 1.

c. 1.7808

d. 0.739

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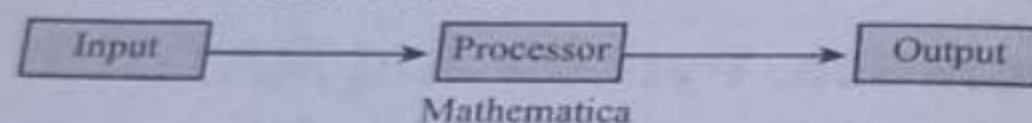
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Python was conceptualized by Dutch Programmer Guido Van Rossum and released in 1991. It is a high level programming language. The website to download and install Python is www.python.org. We can install and run it for free.

Mathematica

First of all, install and run Mathematica in your computer. The details of running Mathematica differ from one computer to another computer, but the calculations is same in all cases.

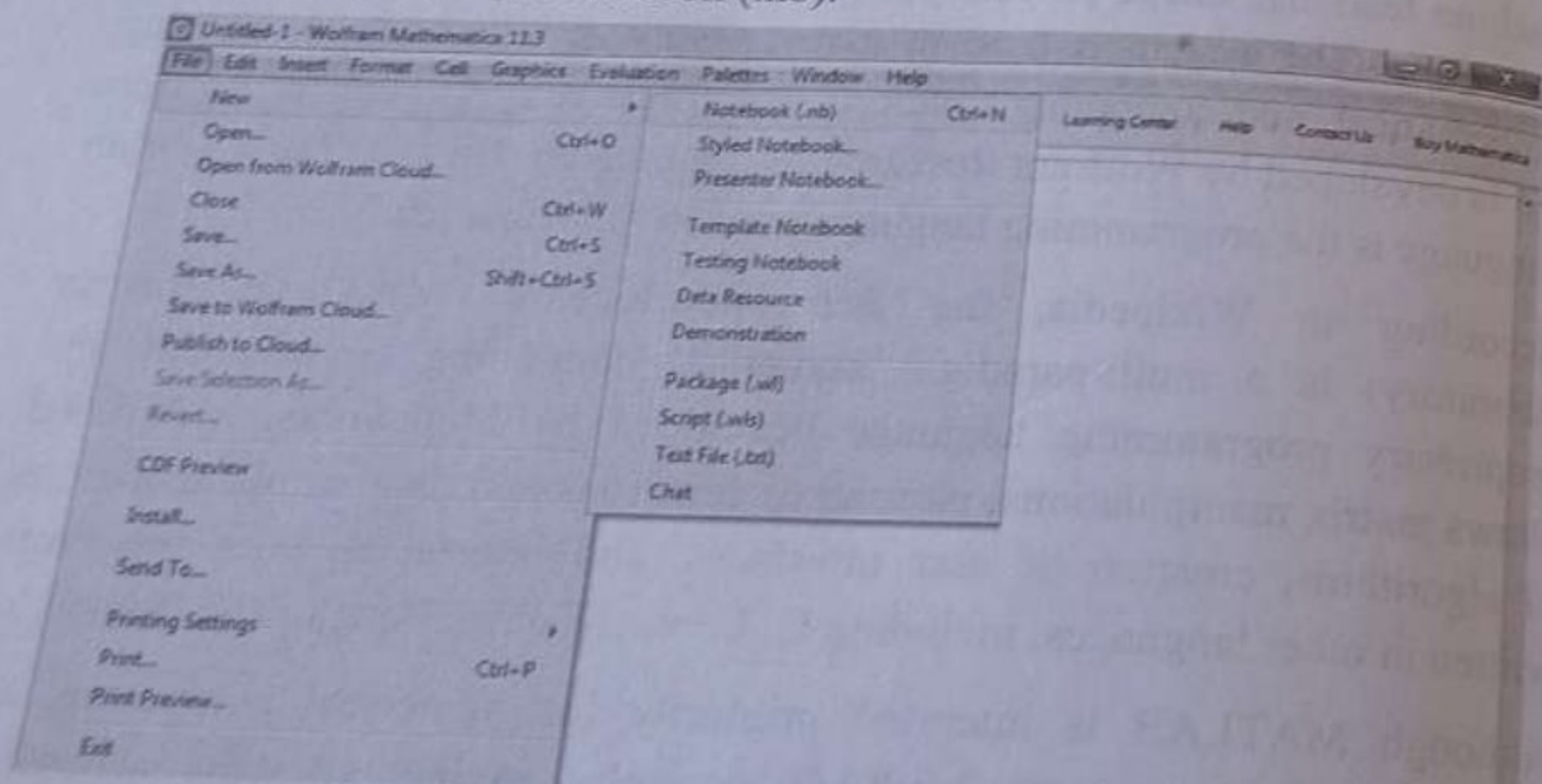


Mathematica is the creation of Stephen Wolfram, a theoretical physicist who made the important contribution to Mathematics. Wolfram describes Mathematica as "the world's only fully integrated environment for technical computing." It is a software package that does algebraic manipulations as well as the graphics. It is the world's fully integrated environment for technical computing.

Mathematica Notebook

When we start Mathematica, we get a menu bar at the top of the screen. To get started computing with Mathematica, we need to create a Notebook. For this, from the menu bar select

File $\Rightarrow$ New $\Rightarrow$ Notebook (.nb).



When we enter a command then the Mathematica Kernel (the part of the software that actually does the computation) executes it and returns the result. For example

In[1] := 1 + 3

Out [1] = 4

The input cell (labeled by In[1] :=) contains the expression 1 + 3, which Mathematica evaluates and gives the result 4 in the output cell (labeled by Out[1] =).

Here, we have to type only '1 + 3'; then Mathematica automatically supplied the label "In[1] := " and same for output. If we see far right of the document, we see the brackets that indicate the grouping of the material into cells.

To Run Mathematica

1. When Mathematica starts up, it gives a blank notebook.
2. Enter the input into the notebook.
3. Type Shift + Enter for output [To type Shift + Enter, hold down the shift key, then press Enter]

When we send Mathematica input from notebook, it will label input with In[n] := and corresponding output with Out[n] =. To exist Mathematica, we can choose the Quit menu item in the notebook interface.

- Note: (i) The arguments of Mathematica functions should be enclosed by square [] brackets.
- (ii) The built-in functions should begin with CAPITAL LETTERS.

Arithmetic Operations in Mathematica

Symbols	Operations
+	add
-	minus
space or *	multiply
/	divide
^	power

The Four Kinds of Bracket in Mathematica

(term)	parentheses for grouping
f[x]	square brackets for functions
{a, b, c}	curly braces for list
a[[i]]	double brackets for indexing (Part[a, i])

Using Mathematica, we can calculate exact value of expression which has more than 10 (in general electronic calculator) places after decimal. Mathematica can handle numbers of any size.

End you input with // N.

It gives approximate value.

For Examples

In[2]:= 2^50

Out[2]= 1125899906842624

In[3]:= 2^50//N

Out[3]= 1.1259×10¹⁵

Some Mathematical Functions

Sqrt[x]	$\sqrt{x}$
Exp[x]	e^x
Log[x]	$\log_e x$
Log[a, x]	$\log_a x$
Sin[x], Cos[x], Tan[x]	$\sin x, \cos x, \tan x$
Abs[x]	x , absolute value
Mod[m, n]	m modulo n
Pi	π
E	e

For Previous Results

% the last result

% % the next to last result

% % ... % (n times) the nth previous result.

Note: x = value assign a value to the variable x.

x = . or clear [x] remove any value assigned to x.

Relational and Logical Operations

x == y	equal
x != y or x ≠ y	unequal
x > y	greater than
x >= y or x ≥ y	greater than or equal to
x < y	less than
x <= y or x ≤	less than or equal to

Graphing

To plot function f from $x_{\min}$ to $x_{\max}$, the basic syntax isPlot [f[x], {x, $x_{\min}$, $x_{\max}$ }]

Redrawing and Combining Plots

Show[Plot]

redraw a plot

Show[Plot, Option → Value]

redraw with options changed

Show[Plot1, Plot2, ...]

combine several plots

Plotting List of Data

ListPlot [{y₁, y₂, ...}]ListPlot [{x₁, y₁}, {x₂, y₂}, ...]

ListPlot [List, PlotJoined → True] join the points with lines.

Limit and Continuity

We compute limit in mathematica in the following ways.

Limit [f [x], x → a]

Examples:

In[9]:= Limit [2 * x + 3, x → 1]

Out[9]= 5

In[10]:= Limit [(7 * x^2 + 4 * x) / x, x → 0]

Out[10]= 4

In[11]:= Limit [(√x - 3) / (x - 9), x → 9]

Out[11]= $\frac{1}{6}$

In[12]:= Limit [(x - √(8 - x^2)) / (√(x^2 + 12) - 4), x → 2]

Out[12]= 4

In[13]:= Limit [(x^8 - 256) / (x - 2), x → 2]

Out[13]= 1024

In[1]:= Limit [(x^2 + 7 * x + 3) / (9 * x^2 + 7 * x + 2), x → ∞]

Out[1]= $\frac{1}{9}$

In[2]:= Limit [(√(x - 1) - √x), x → ∞]

Out[2]= 0

In[3]:= Limit [(√(x^2 + 1)) / (x + 1), x → ∞]

Out[3]= 1

Limit of trigonometric functions

In[4]:= Limit[Sin[x] / x, x → 0]

Out[4]= 1

In[5]:= Limit[Sin[4 x] / Sin[5 x], x → 0]

Out[5]= $\frac{4}{5}$

In[6]:= Limit[(Tan[3 * x] - 2 * x) / (3 * x - Sin[2 * x]), x → 0]

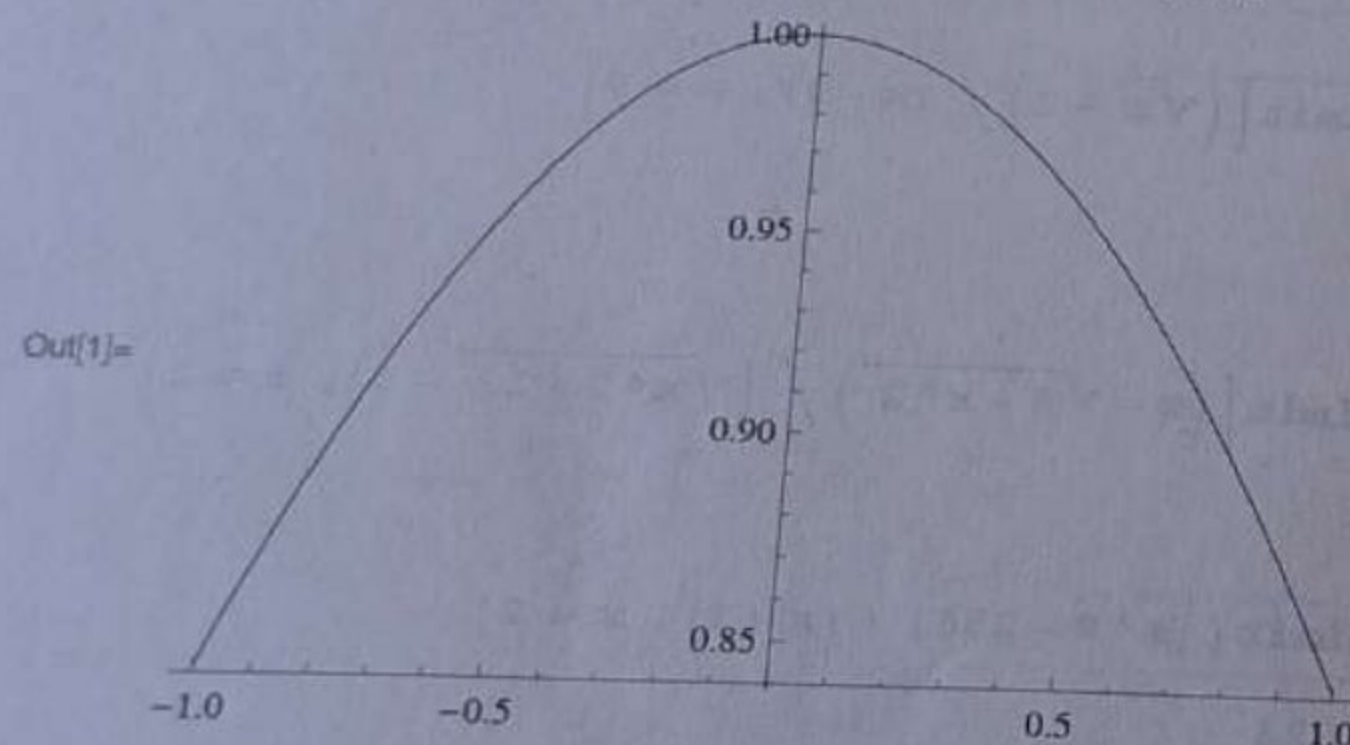
Out[6]= 1

In[7]:= Limit[(Sin[x] - Cos[x]) / (x - π / 4), x → π / 4]

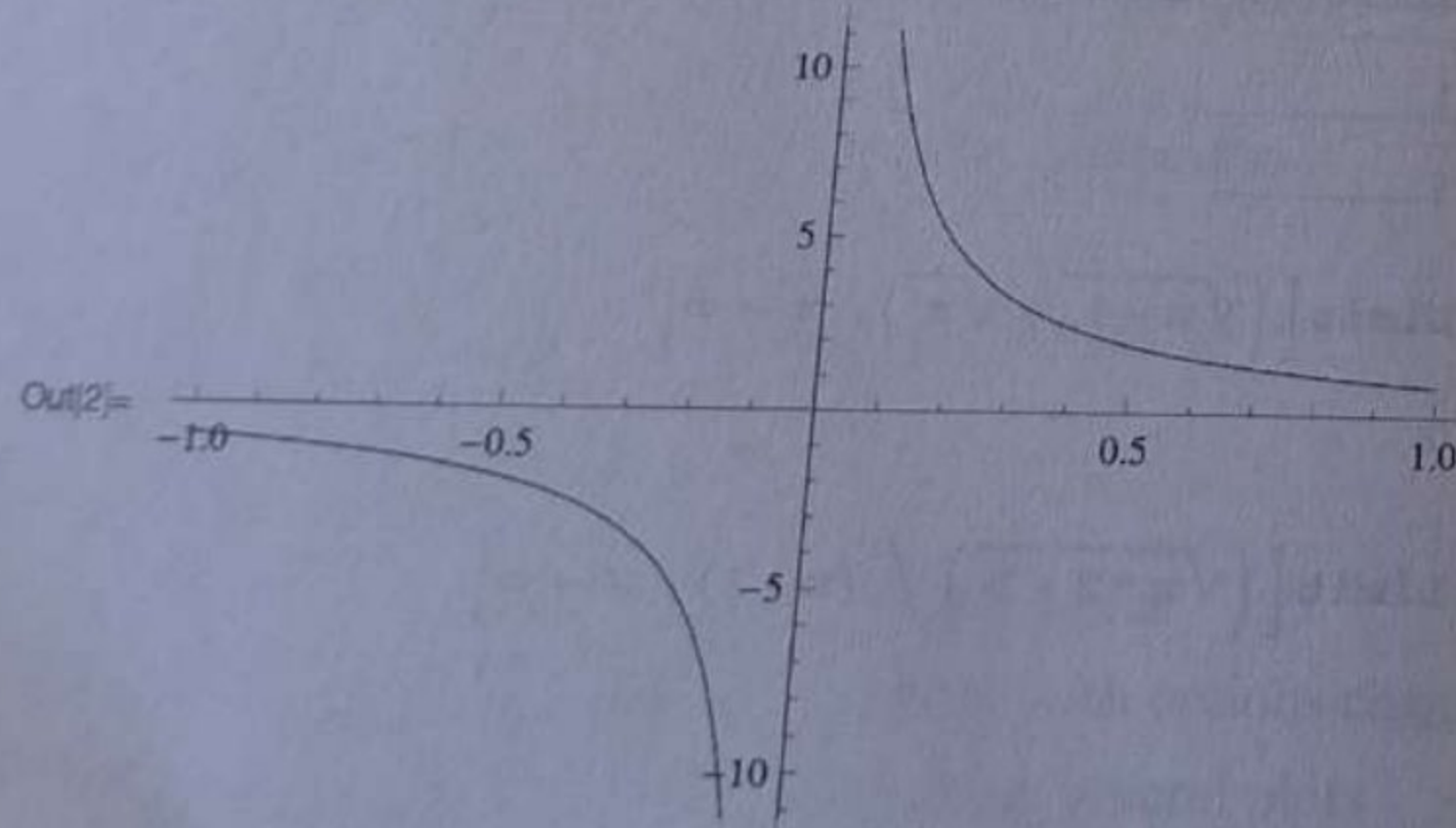
Out[7]= $\sqrt{2}$

Some graphs

In[1]:= GraphicsRow[{Plot[Sin[x] / x, {x, -1, 1}]}]



In[2]:= GraphicsRow[{Plot[Sin[x] / x^2, {x, -1, 1}]}]



One sided limits

In[11]:= (*From the left*) Limit[Sin[x] / x^2, x → 0, Direction → 1]

Out[11]= -∞

In[4]:= (*From the left*) Limit[x / Abs[x], x → 0, Direction → 1]

Out[4]= -1

In[5]:= (*From the right*) Limit[x / Abs[x], x → 0, Direction → -1]

Out[5]= 1

In[6]:= (* from the left*) Limit [sin[x]/x^2,
x → 0, Direction → 1]

= -∞

Out[6] = (*from the right*) Limit [sin[x]/x^2,
x → 0, Direction → -1]

= ∞

Differentiation

Mathematica allows us to enter derivatives in input form as $D[f[x], x]$, or in standard form as $\partial_x f[x]$.

Higher-order derivatives are specific as $D[f[x], \{x, n\}]$ where 'n' is 2 for the second derivative, 3 for the third and so on. In standard form the second derivative can be entered as $\partial(x, 2) f[x]$.

In [1]; = D[sin [x], {x, 2}].

Out [1] = -sin [x]

Mathematica also recognizes prime notation. For example

In [2]: = {sin'[x], cos'[x]}

Output [2] = {cos[x], -sin[x]}

Implicit Function

Implicit Function = $x^2 + y^2 = 1$;

Solve [D[implicit Function, x], y'[x]]

Some examples:

In[14]:= D[2 * x + 5, x]

Out[14]= 2

In[15]:= D[Sqrt [x], x]

Out[15]= $\frac{1}{2\sqrt{x}}$

$$\text{In[17]} = D[7 * x^{10} + 5 * x^3 - 6, x]$$

$$\text{Out[17]} = 15x^2 + 70x^9$$

$$\text{In[18]} = D[(x^2 + 5 * x) (3 * x^2 - x), x]$$

$$\text{Out[18]} = (-1 + 6x)(5x + x^2) + (5 + 2x)(-x + 3x^2)$$

$$\text{In[20]} = D[\sin[x], x]$$

$$\text{Out[20]} = \cos[x]$$

$$\text{In[21]} = D[\cos[x], x]$$

$$\text{Out[21]} = -\sin[x]$$

$$\text{In[22]} = D[\tan[x], x]$$

$$\text{Out[22]} = \sec[x]^2$$

$$\text{In[24]} = D[\csc[x], x]$$

$$\text{Out[24]} = -\cot[x] \csc[x]$$

$$\text{In[25]} = D[\sec[x], x]$$

$$\text{Out[25]} = \sec[x] \tan[x]$$

$$\text{In[27]} = D[\exp[x], x]$$

$$\text{Out[27]} = e^x$$

$$\text{In[28]} = D[\log[x], x]$$

$$\text{Out[28]} = \frac{1}{x}$$

$$\text{In[30]} = \{\sin'[x], \cos'[x]\}$$

$$\text{Out[30]} = \{\cos[x], -\sin[x]\}$$

Application of Derivatives

Maxima and Minima

Maximize $[f[x], x]$

Minimize $[f[x], x]$

Some examples:

$$\text{In[1]} = \text{Minimize}[3 * x^2 - 6 * x + 4, x]$$

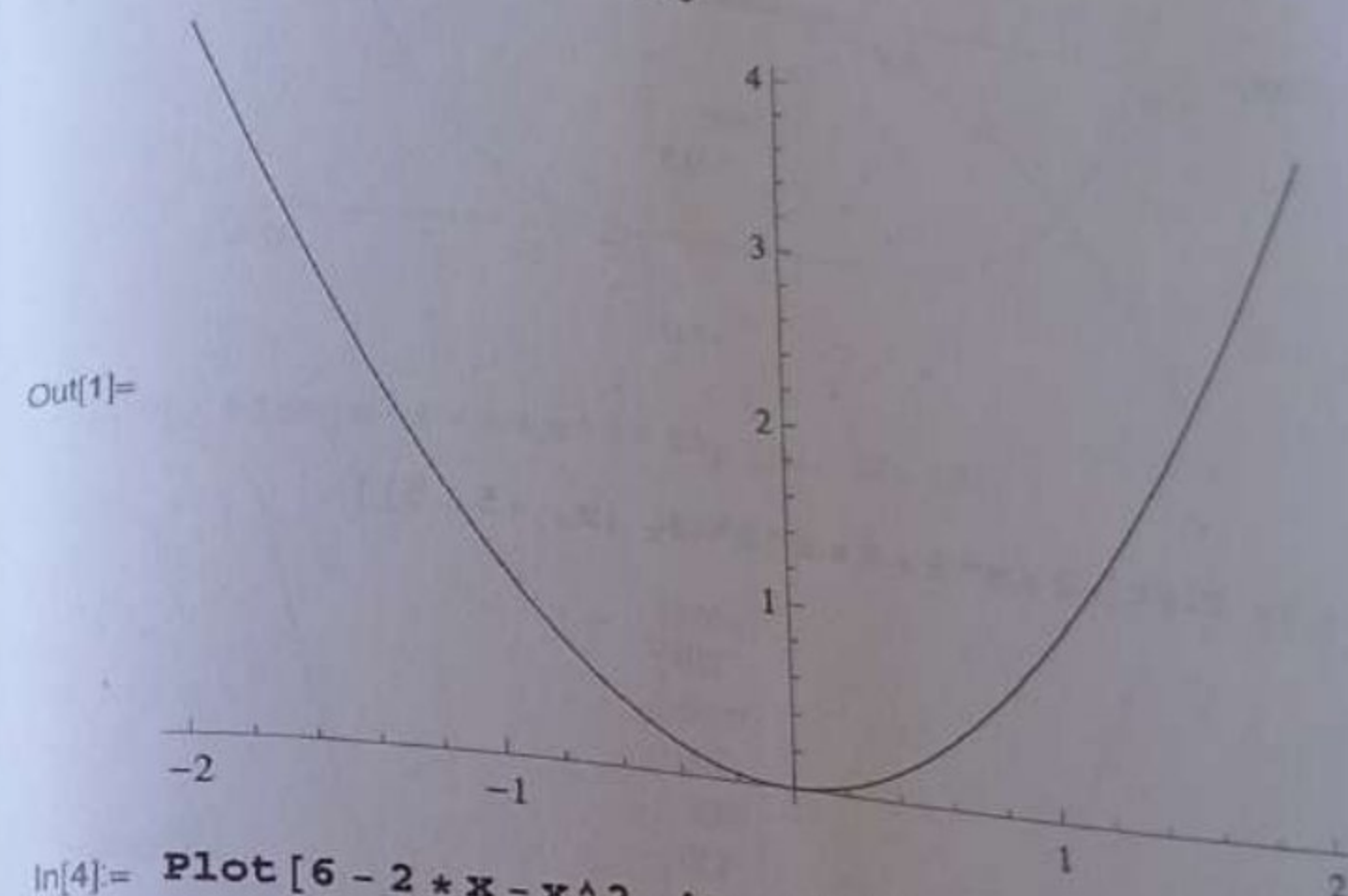
$$\text{Out[1]} = \{1, \{x \rightarrow 1\}\}$$

$$\text{In[2]} = \text{Maximize}[-3 * x^2 - 6 * x + 4, x]$$

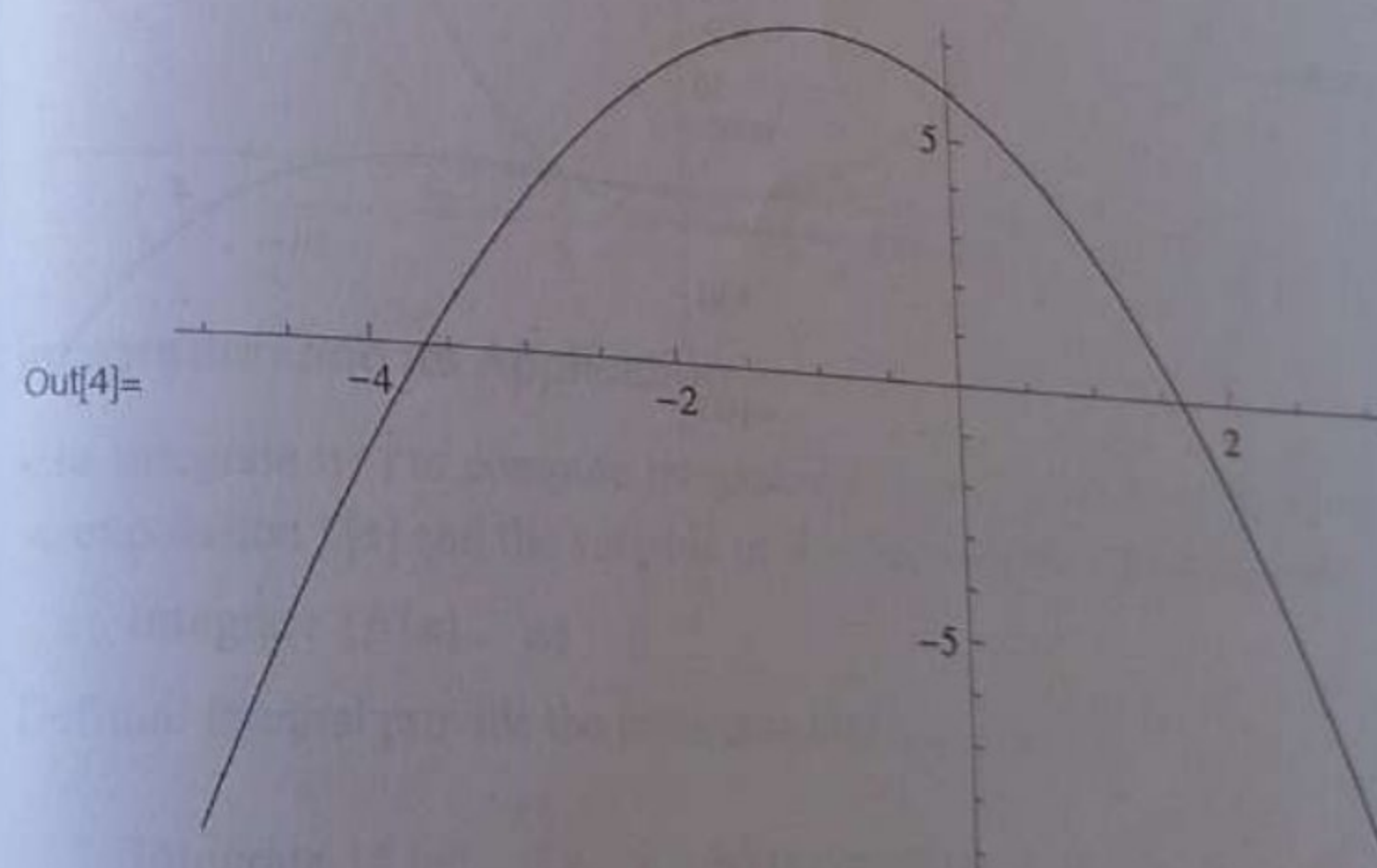
$$\text{Out[2]} = \{7, \{x \rightarrow -1\}\}$$

Some graphs

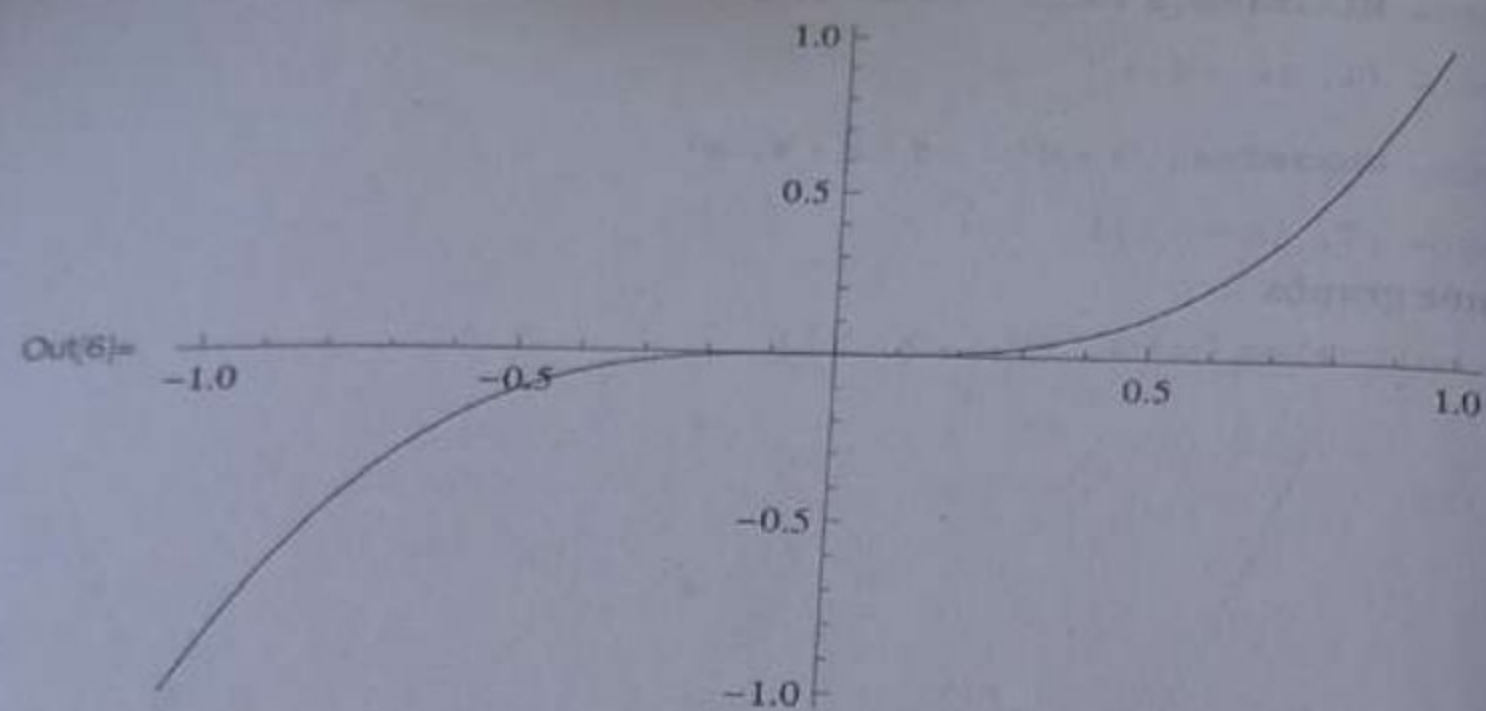
$$\text{In[1]} = \text{Plot}[x^2, \{x, -2, 2\}]$$



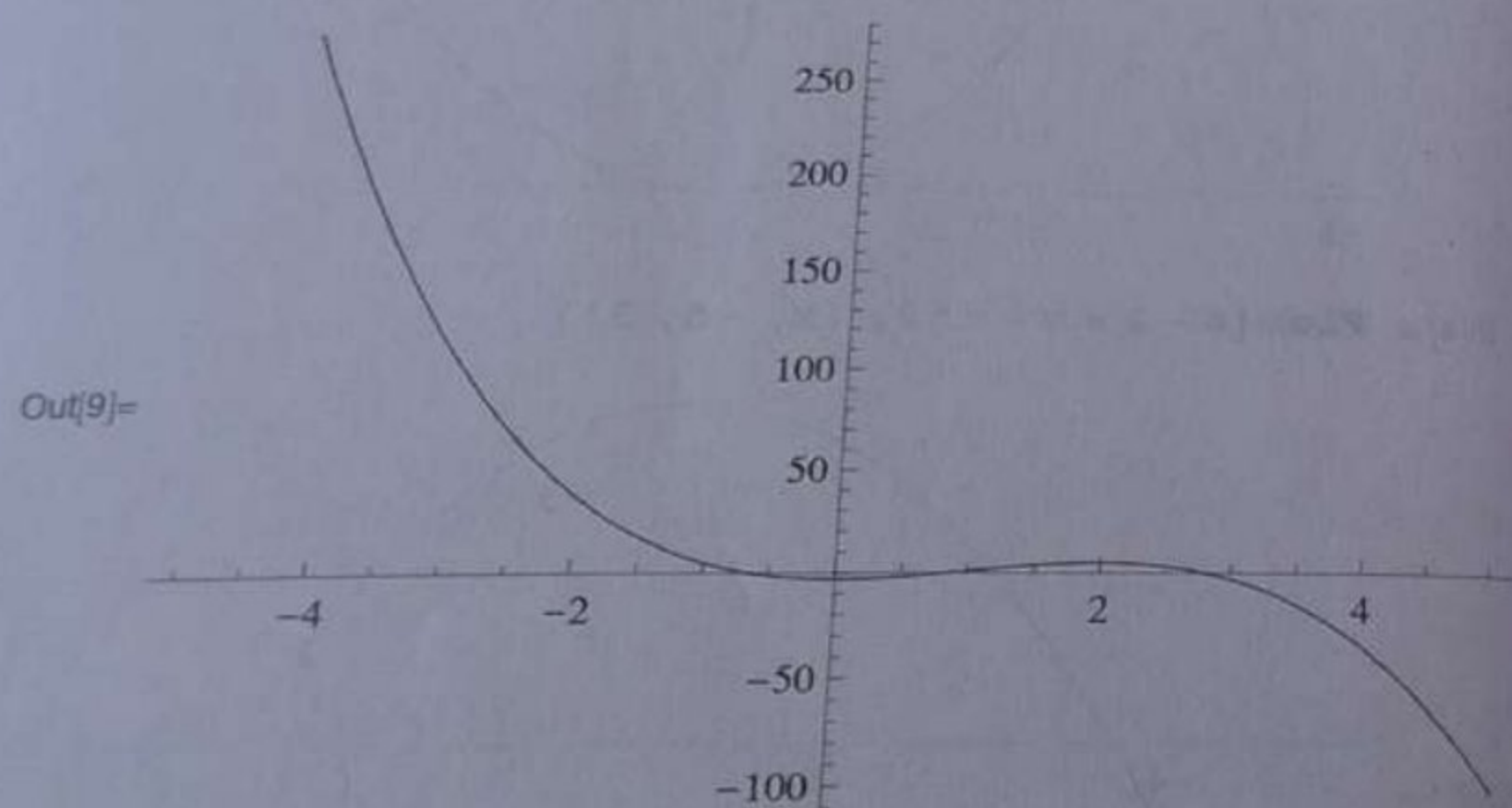
$$\text{In[4]} = \text{Plot}[6 - 2 * x - x^2, \{x, -5, 3\}]$$



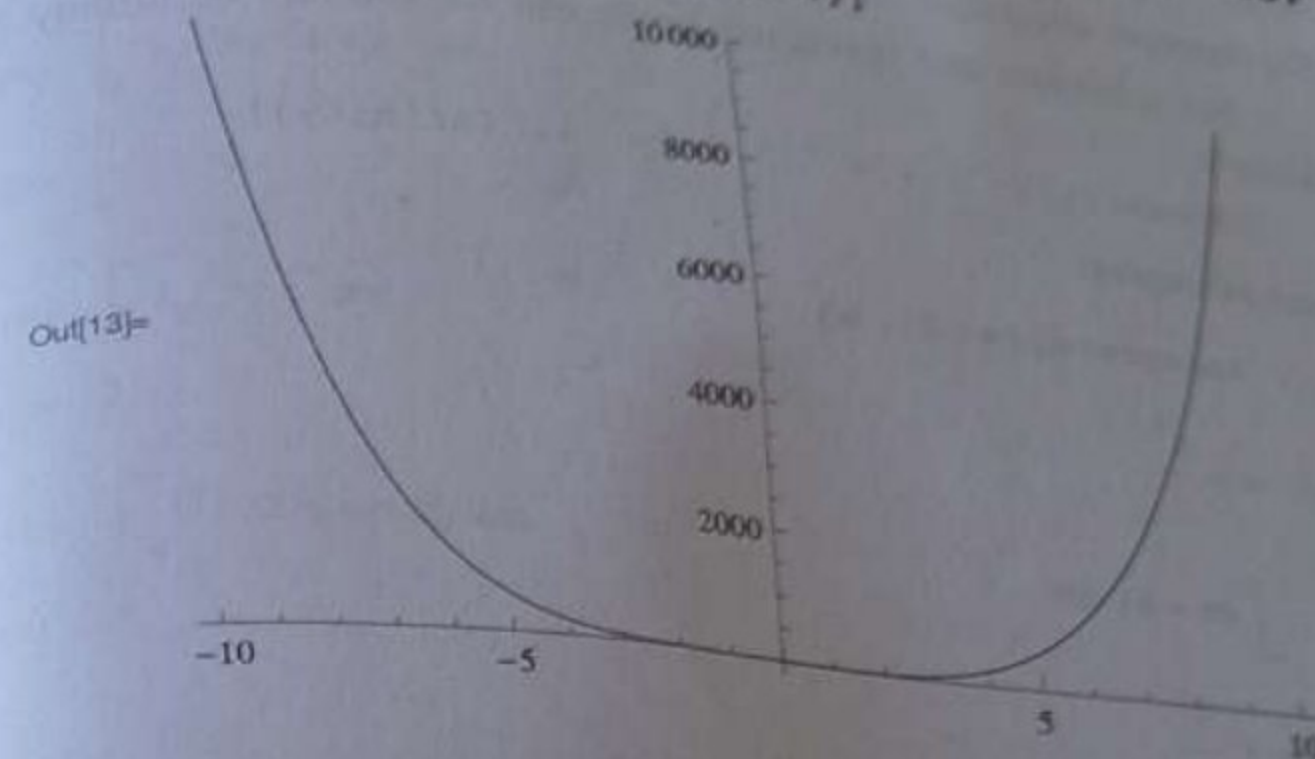
In[6]= Plot[x^3, {x, -1, 1}]



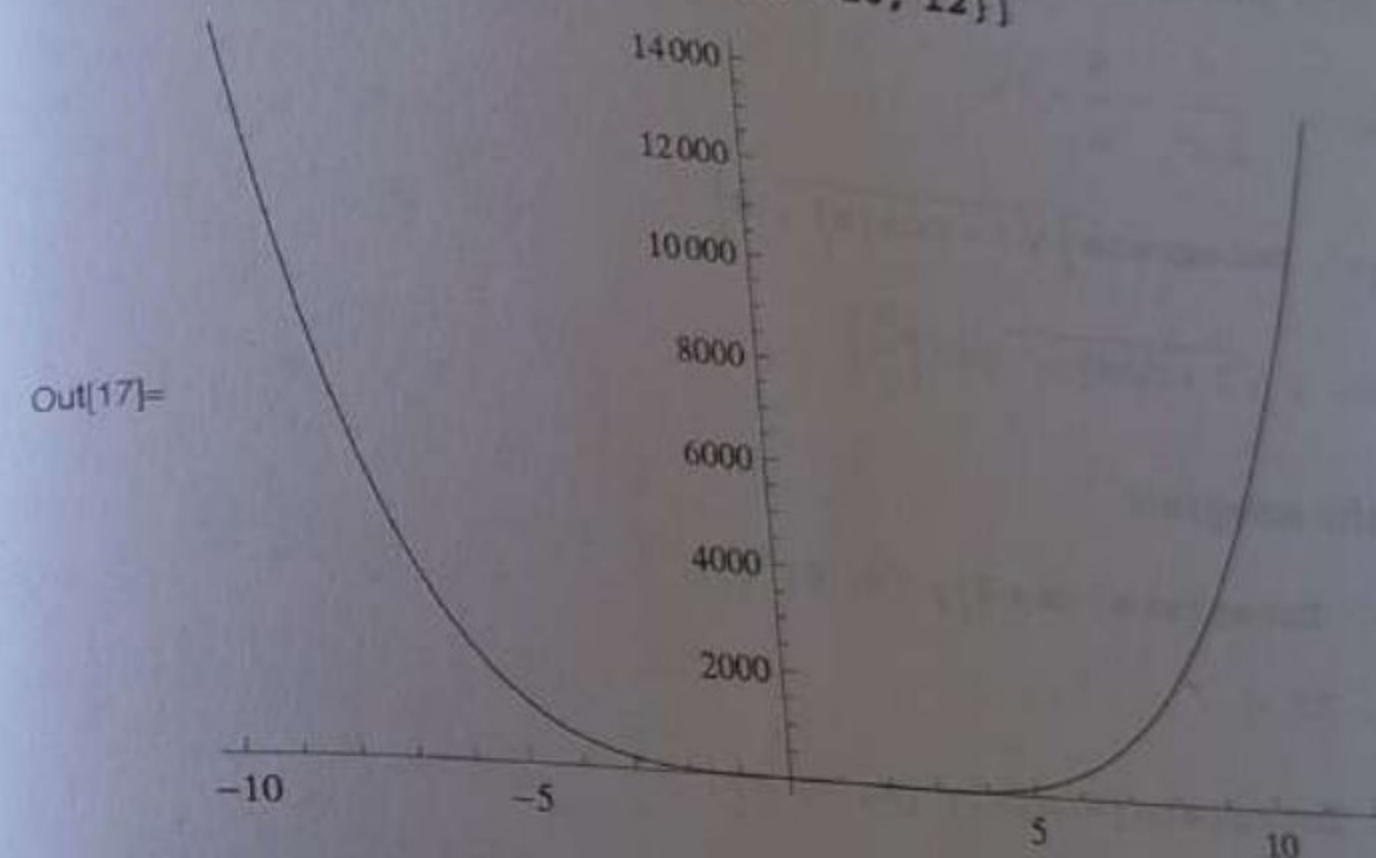
In[9]= Plot[-2 * x^3 + 6 * x^2 - 3, {x, -5, 5}]



In[13]= Plot[x^4 - 2 * x^2, {x, -10, 10}]



In[17]= Plot[x^4 - 4 * x^3 + 10, {x, -10, 12}]



Integration and Its Application

Use Integrate or $\int$ to compute integrations. In indefinite integrals we specify an expression $f[x]$ and the variable of integration in the following form

Integrate [$f[x]$, x]

Definite integral provide the minimum and maximum limits.

Integrate [$f[x]$, { x , a , b }] to evaluate $\int_a^b f(x) dx$

For improper integral,

The minimum and maximum limits can be $-\infty$ or ∞ . For example:

Integrate $[1/(x^3 + x^3), \{x, 1, \text{Infinity}\}]$

Some examples:

In[1]:= Integrate $[(x+1), x]$

$$\text{Out[1]} = x + \frac{x^2}{2}$$

$$\text{In[2]} = \int (x+1) dx$$

$$\text{Out[2]} = x + \frac{x^2}{2}$$

In[1]:= Integrate $[(3 * x^3 + 4 * x + 5) / x^3, x]$

$$\text{Out[1]} = -\frac{5}{2x^2} - \frac{4}{x} + 3x$$

In[2]:= Integrate $[\sqrt{1 + \cos[x]}, x]$

$$\text{Out[2]} = 2 \sqrt{1 + \cos[x]} \tan\left[\frac{x}{2}\right]$$

Definite integrals

In[3]:= Integrate $[(x+4), \{x, 0, 2\}]$

$$\text{Out[3]} = 10$$

In[4]:= Integrate $[\sin[x], \{x, 0, \pi\}]$

$$\text{Out[4]} = 2$$

In[5]:= Integrate $[2 * x / (x^2 + 1), \{x, 0, 1\}]$

$$\text{Out[5]} = \log[2]$$

In[7]:= Integrate $[\sqrt{16 - x^2}, \{x, 0, 4\}]$

$$\text{Out[7]} = 4\pi$$

Improper integrals

Appendix

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$$\text{In[9]} = \int_0^{\infty} 1 / (x^2 + 1) dx$$

$$\text{Out[9]} = \frac{\pi}{2}$$

$$\text{In[10]} = \int_0^1 1 / \sqrt{x} dx$$

$$\text{Out[10]} = 2$$

$$\text{In[11]} = \int_{-\infty}^{\infty} 1 / (1 + x^2) dx$$

$$\text{Out[11]} = \pi$$

loading

Python

After installation of Python, when we open IDLE, Python Shell, we will see >>> prompt inside the Python window. We can run Python codes in IDLE, VS Code, Jupiter Notebook. For example,

```
>>> (3+2)
```

Press **Enter** to get the output 5.

Next, `print("Hello, I am a BCA student")`

Press **Enter** for output.

Output will be,

Hello, I am a BCA student.

You can save file by clicking on file and then save or if you have already created the file then do save as.

NOTE The Python codes are not unique. The same task can be written in many different ways.

The Python uses () bracket for input. For output, we can use `print()`.

The main areas that we have to know for Python are

1. Syntax and spacing
2. Variables and data types
3. Input and print functions
4. Conditional statements
5. For and while loops

Basic Operators

```
print(a+b) #add
print(a-b) #subtract
print(a*b) #multiply
print(a/b) #divide (float)
print(a//b) #divide
(integer)
print(a%b) #remainder
print(a**b) #power
```

Function

Square root
Cube root
Exponential
Logarithm
Sine

Comparison Python Code

```
Equal to (==)
Not equal to (!=)
Greater than (>)
Less than (<)
Greater than or equal to
(>=)
Less than or equal to (<=)
```

Python Code

```
Sqrt(x)
x**(1/3)
exp(x)
log(x)
sin(x)
```

The easiest way to solve calculus Python is by using SymPy. First of all, you have to install it. You can do it online using SymPy Live Shell or Google Colab. The following is the summary table for Python Code of limits, derivatives, and antiderivatives.

Concept	Symbol	Python Command
Variable	x	<code>x = sp.symbols('x')</code>
Limit	$\lim_{x \rightarrow a} f(x)$	<code>sp.limit(f, x, a)</code>
Derivative	$\frac{d}{dx} f(x)$	<code>sp.diff(f, x)</code>
Partial derivative	$\frac{\partial}{\partial x} f(x, y)$	<code>sp.diff(f, x)</code>
	$\frac{\partial}{\partial y} f(x, y)$	<code>sp.diff(f, y)</code>
Indefinite integral	$\int f(x) dx$	<code>sp.integrate(f, x)</code>
Definite integral	$\int_a^b f(x) dx$	<code>sp.integrate(f, (x, a, b))</code>

Note: We have to write infinity as `sp.oo`

Examples

1. Evaluate $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5}$

```
import sympy as sp
```

```
# Define symbol
```

```
x = sp.symbols('x')
```

```
# Define expression
```

```
expr = (x**2 - 25) / (x - 5)
```

```
# Compute limit
```

```
result = sp.limit(expr, x, 5)
```

```
# Print result
```

```
print(result)
```

10

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               (integer)
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print(a**b) #power
```

Function

Square root
Cube root
Exponential
Logarithm
Sine

Comparison Python Code

```
Equal to (==)
Not equal to (!=)
Greater than (>)
Less than (<)
Greater than or equal to
               (>=)
Less than or equal to (<=)
```

Python Code

```
Sqrt(x)
x**(1/3)
exp(x)
log(x)
sin(x)
```

The easiest way to solve calculus Python is by using SymPy. First of all, you have to install it. You can do it online using SymPy Live Shell or Google Colab. The following is the summary table for Python Code of limits, derivatives, and antiderivatives.

Concept	Symbol	Python Command
Variable	x	<code>x = sp.symbols('x')</code>
Limit	$\lim_{x \rightarrow a} f(x)$	<code>sp.limit(f, x, a)</code>
Derivative	$\frac{d}{dx} f(x)$	<code>sp.diff(f, x)</code>
Partial derivative	$\frac{\partial}{\partial x} f(x, y)$	<code>sp.diff(f, x)</code>
	$\frac{\partial}{\partial y} f(x, y)$	<code>sp.diff(f, y)</code>
Indefinite integral	$\int f(x) dx$	<code>sp.integrate(f, x)</code>
Definite integral	$\int_a^b f(x) dx$	<code>sp.integrate(f, (x, a, b))</code>

Note: We have to write infinity as `sp.oo`

Examples

1. Evaluate $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5}$

```
import sympy as sp

# Define symbol
x = sp.symbols('x')

# Define expression
expr = (x**2 - 25) / (x - 5)

# Compute Limit
result = sp.limit(expr, x, 5)

# Print result
print(result)
```

10

2. Evaluate $\lim_{x \rightarrow y} \frac{\sin x - \sin y}{x - y}$

```
import sympy as sp
```

```
# Define symbols
```

```
x, y = sp.symbols('x y')
```

```
# Define expression
```

```
expr = (sp.sin(x) - sp.sin(y)) / (x - y)
```

```
# Compute limit
```

```
result = sp.limit(expr, x, y)
```

```
# Print result
```

```
print(result)
```

```
cos(y)
```

3. Find $\frac{dy}{dx}$ if $y = 7x^{10} + 5x^3 - 6$

```
import sympy as sp
```

```
x = sp.symbols('x')
```

```
expr = 7*x**10 + 5*x**3 - 6
```

```
result = sp.diff(expr, x)
```

```
print(result)
```

```
70*x**9 + 15*x**2
```

4. If $f(x, y) = x^2 + 3xy + 2y^2$ then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

```
import sympy as sp
```

```
# Define symbols
```

```
x, y = sp.symbols('x y')
```

```
# Define function
```

```
f = x**2 + 3*x*y + 2*y**2
```

```
# Partial derivatives
```

```
fx = sp.diff(f, x)
```

```
fy = sp.diff(f, y)
```

```
# Print results
```

```
print("fx =", fx)
```

```
print("fy =", fy)
```

```
fx = 2*x + 3*y
```

```
fy = 3*x + 4*y
```

5. Evaluate $\int (x+1) dx$

```
import sympy as sp
```

```
# Define symbol
```

```
x = sp.symbols('x')
```

```
# Define expression
```

```
expr = x + 1
```

```
# Indefinite integral
```

```
result = sp.integrate(expr, x)
```

```
# Print result
```

```
print(result)
```

```
x**2/2 + x
```

Loading

6. Evaluate $\int_0^{\frac{\pi}{2}} x \sin x \, dx$.

```
import sympy as sp

# Define symbol
x = sp.symbols('x')

# Define expression
expr = x * sp.sin(x)

# Definite integral from 0 to pi/2
result = sp.integrate(expr, (x, 0, sp.pi/2))

# Print result
print(result)

1
```

MATLAB

Appendix

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Loading

MATLAB is an interactive matrix based system for scientific and engineering numerical computation as well as visualization. MATLAB is developed by Mathworks. It is high level programming language. When the MATLAB software is started, a window is opened. The main part is the Command Window.

In Command Window, we see

```
>>
```

It is called the prompt. In Command Window, if we enter any MATLAB command or expression then MATLAB will immediately respond with result.

To get out of MATLAB, either type quit at the prompt or choose file then Exit MATLAB from the menu.

In addition to the Command Window, there are several other windows in MATLAB. Directly above the Command Window, there is a pull down Directory for Current Directory. The folder which is set as Current Directory is that where files will be saved. It is the work directory by default, but it can be changed.

To the left of the Command Window, there are two tabs; one for Current Directory Window and other for Workspace Window. If we choose Current Directory Tab, then the files stored in that directory are displayed. The Command History Window shows commands that have been entered.

In command window of MATLAB, we see the command prompt

```
>>
```

For example if we type

```
>> 15 + 12
```

MATLAB will display (Press enter for result)

```
ans =
```

```
27
```


6. Evaluate $\int_0^{\pi/2} x \sin x \, dx$.

```
import sympy as sp

# Define symbol
x = sp.symbols('x')

# Define expression
expr = x * sp.sin(x)

# Definite integral from 0 to pi/2
result = sp.integrate(expr, (x, 0, sp.pi/2))

# Print result
print(result)

1
```

MATLAB

Appendix

365

Loading

MATLAB is an interactive matrix based system for scientific and engineering numerical computation as well as visualization. MATLAB is developed by Mathworks. It is high level programming language. When the MATLAB software is started, a window is opened. The main part is the Command Window.

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In command window of MATLAB, we see the command prompt

>>

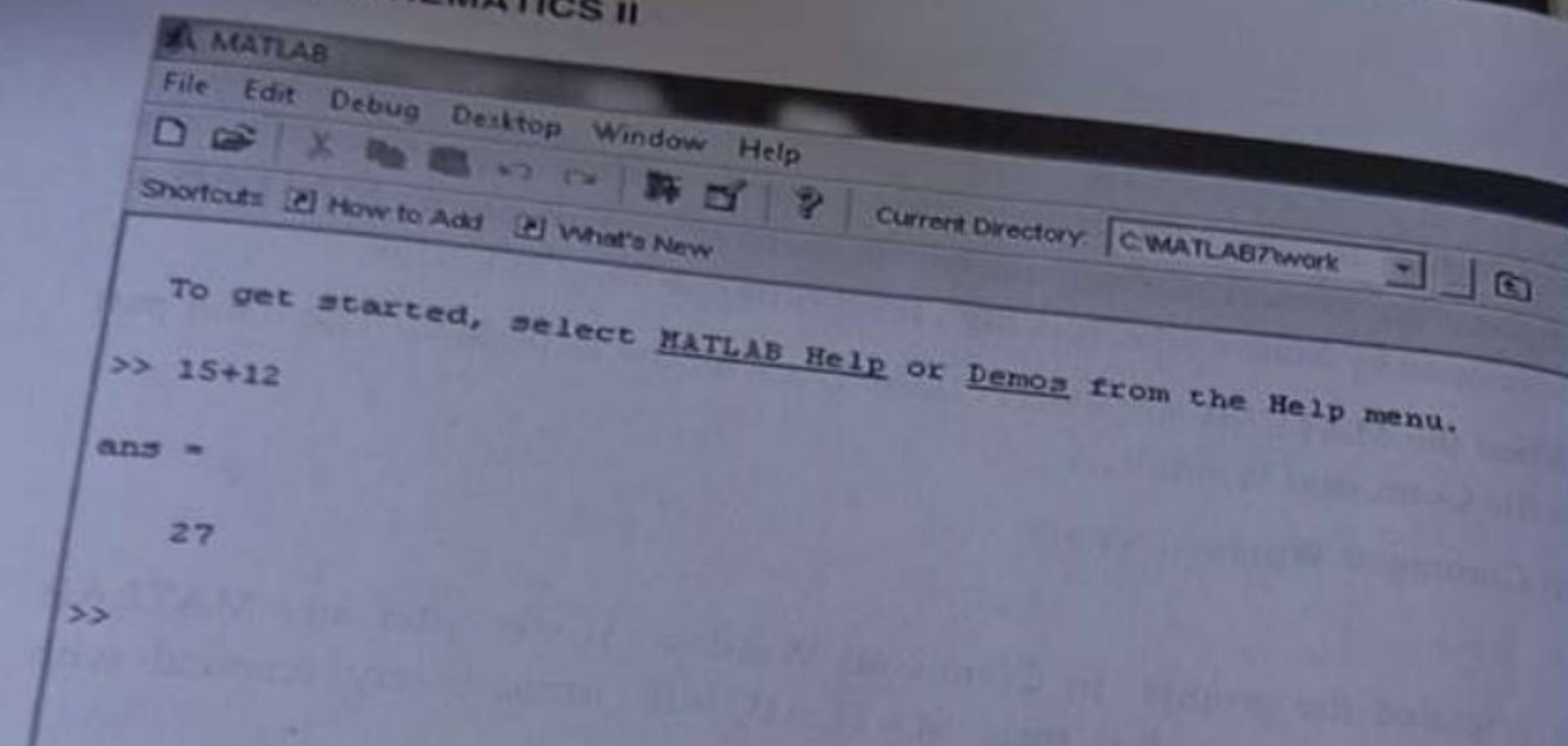
For example if we type

>> 15 + 12

MATLAB will display (Press enter for result)

ans =

27



Next,

```
>> ans - 27 gives the
```

```
ans =
    50
```

Assignment

It refers to assigning values to the variable.

1. Assigning Scalars

Type

```
>> x = 4
```

It will display

```
x =
    4
```

2. Assigning Arrays, Vectors and Matrices

An array is a collection of values by a single variable. One dimensional arrays are called vectors and two dimensional arrays are called matrices.

For example

```
a. >> a = [3 4 5 6]
```

```
a =
    3    4    5    6
```

```
b. >> b = [3; 4; 5; 6]
```

or by transposing

```
>> b = [3 4 5 6]'
```

The result is

```
b =
    3
```

4

5

6

A matrix can be displayed as follows:

```
>> A = [3 4 5; 5 6 7; 7 8 9]
```

```
A =
    3    4    5
    5    6    7
    7    8    9
```

Also, we can separate the rows by the help of Enter key

```
>> A = [3    4    5
        5    6    7
        7    8    9]
```

A(m, n) selects the element in m^{th} row and n^{th} column. For example

```
>> A(1, 2)
ans =
    4
```

Note: There are several built in functions to create matrices.

It is Matrix Laboratory. MATLAB is matrix based system for scientific and engineering numerical computation and visualization. MATLAB is developed by Mathworks.

Arithmetic operation

Operations

Meaning

+	addition
-	subtraction
*	multiplication
\	left division
/	right division
^	power

The difference between left division (\) and right division (/) is as follows:

$$8 / 4 = 2$$

[It means $\frac{8}{4} = 2$]

$$\& \quad 8 \setminus 4 = 0.5000$$

[It means $\frac{4}{8} = 0.5000$]

Built in function

Command	Function
abs ()	absolute value
exp ()	exponential function
sqrt ()	square root
log ()	logarithmic function

Note: By default MATLAB produces 4 digits after decimal places.

If we want move digits after the decimal places, then we should `>> format long`.

If you need, any help, take the help of `>> help format`.

Trigonometric function

Command	Function
<code>sin ()</code>	sine
<code>cos ()</code>	cosine
<code>tan ()</code>	tangent
<code>cot ()</code>	cotangent
<code>sec ()</code>	secant
<code>csc ()</code>	cosecant

Solving Algebraic Equations in MATLAB

We solve the algebraic equations as follows:

Example (1) Solve $x + 3 = 0$

```
>> solve('x + 3 = 0')
```

ans =

-3.

(2) Solve $x^2 - 5x + 6 = 0$

```
>> solve('x^2 - 5x + 6 = 0')
```

ans =

3

2.

(3) Solve $x^3 - 7x^2 + 36 = 0$

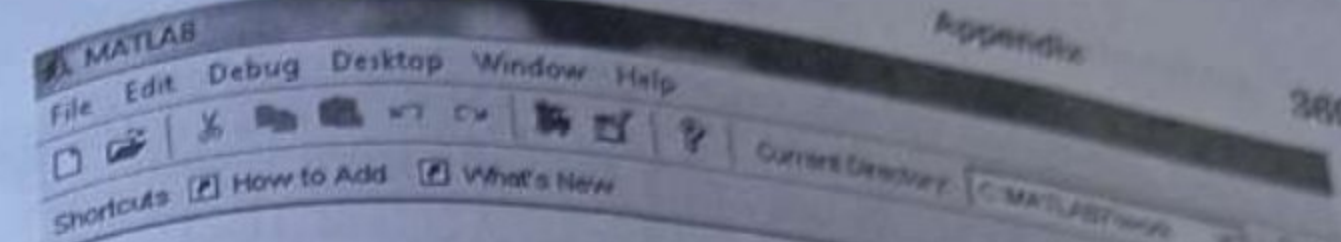
```
>> solve('x^3 - 7*x^2 + 36 = 0')
```

ans =

-2

3

6.



To get started, select **MATLAB Help** or **Demo** from the Help menu.

```
>> solve('x+3=0')
```

ans =

-3

```
>> solve('x^2-5*x+6=0')
```

ans =

3

2

```
>> solve('x^3-7*x^2+36=0')
```

ans =

-2

3

6

Plotting

The plot in MATLAB appears in the Graphic / Figure Window. MATLAB provides facilities for 2-D and 3-D graphics. The most command for 2-D and 3-D plot are `plot` and `plot3` command respectively.

2-D plotting

Syntax: `plot(x data, y data)`

Here, y data are the value of y variable corresponding to x variable which are given at x data, such as $y = f(x)$.

This command plots x data along horizontal axis and y data along vertical axis.

Example:

Plot the function $y = x^2 - 5x + 6$, where $x \in [-5, 10]$.

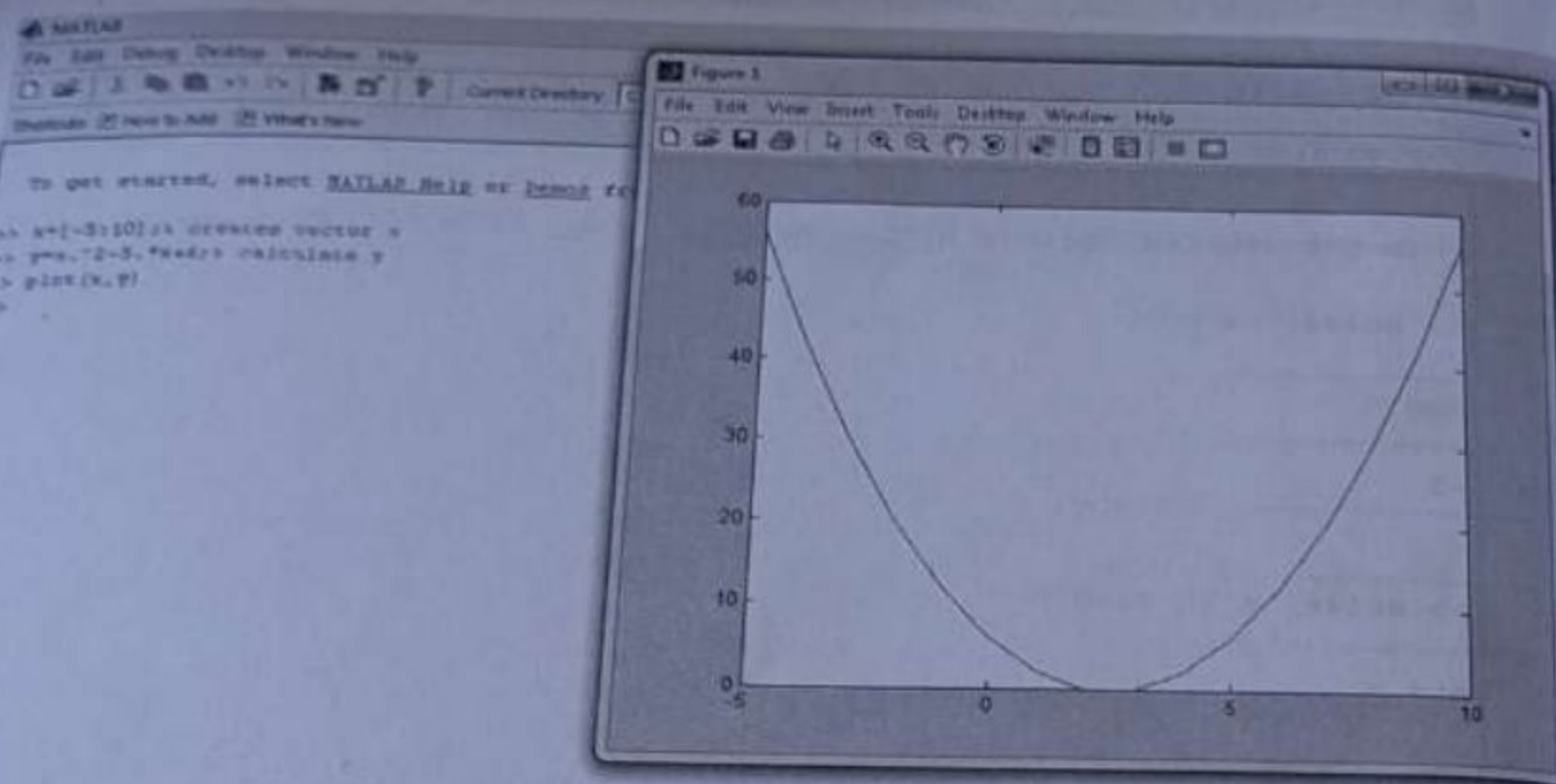
Solution

```
>> x = [-5:10]; % creates vector x
```

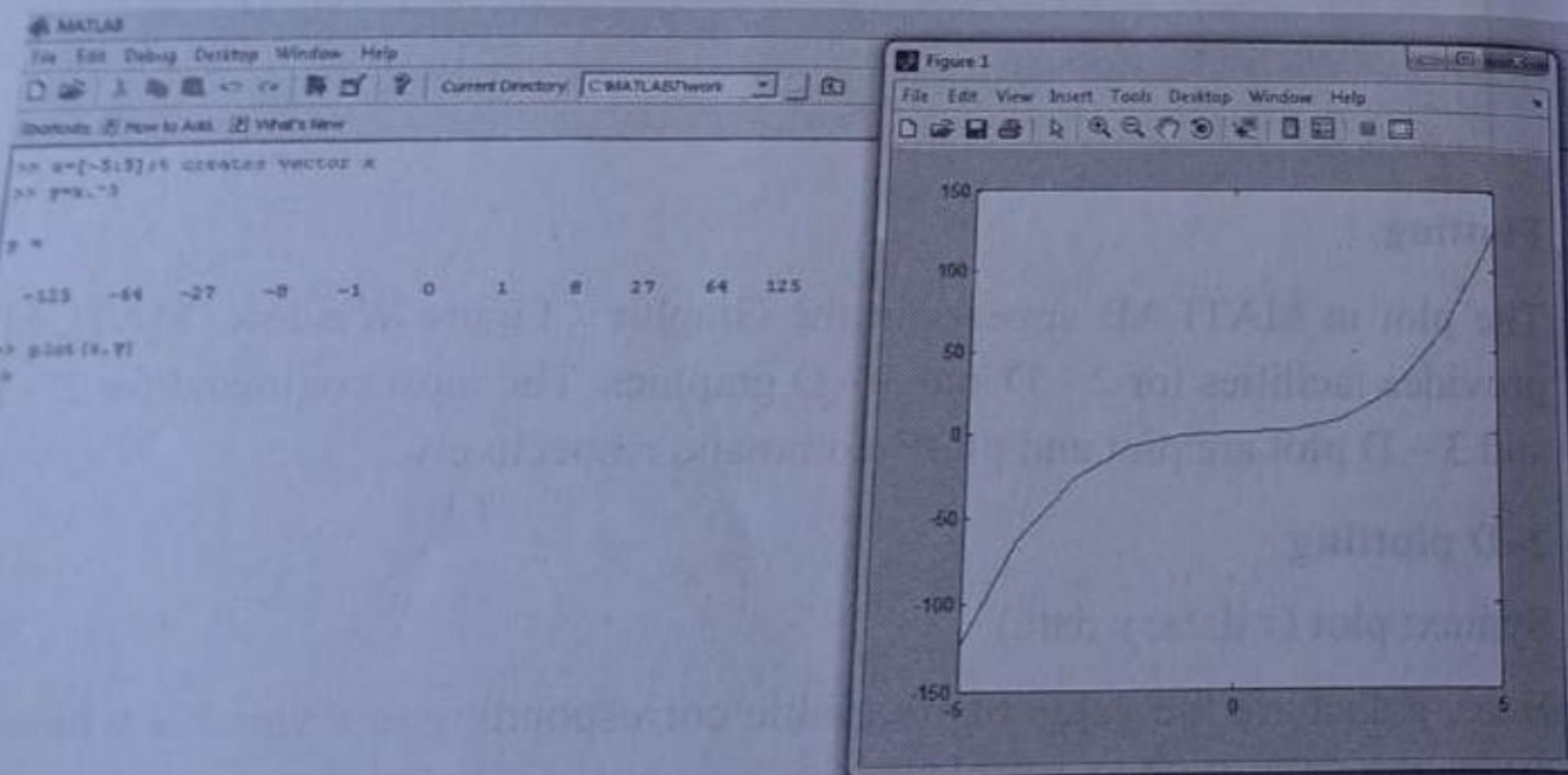
```
>> y = x.^2 - 5.*x + 6; % calculate y
```



```
>> plot (x, y)
```

**Example:**

Plot the function $y = x^3$ in $[-5, 5]$.

Solution**Arrays**

Variable = initial value: step size: final value

Here, initial value is the first element of the row, final value is the last element of the row and step size is the difference between the consecutive elements of that row.

For example

```
>> x = 2:1:10
```

```
x =
```

```
2 4 6 8 10
```

3-D plotting

Syntax: `plot 3(x, y, z)`

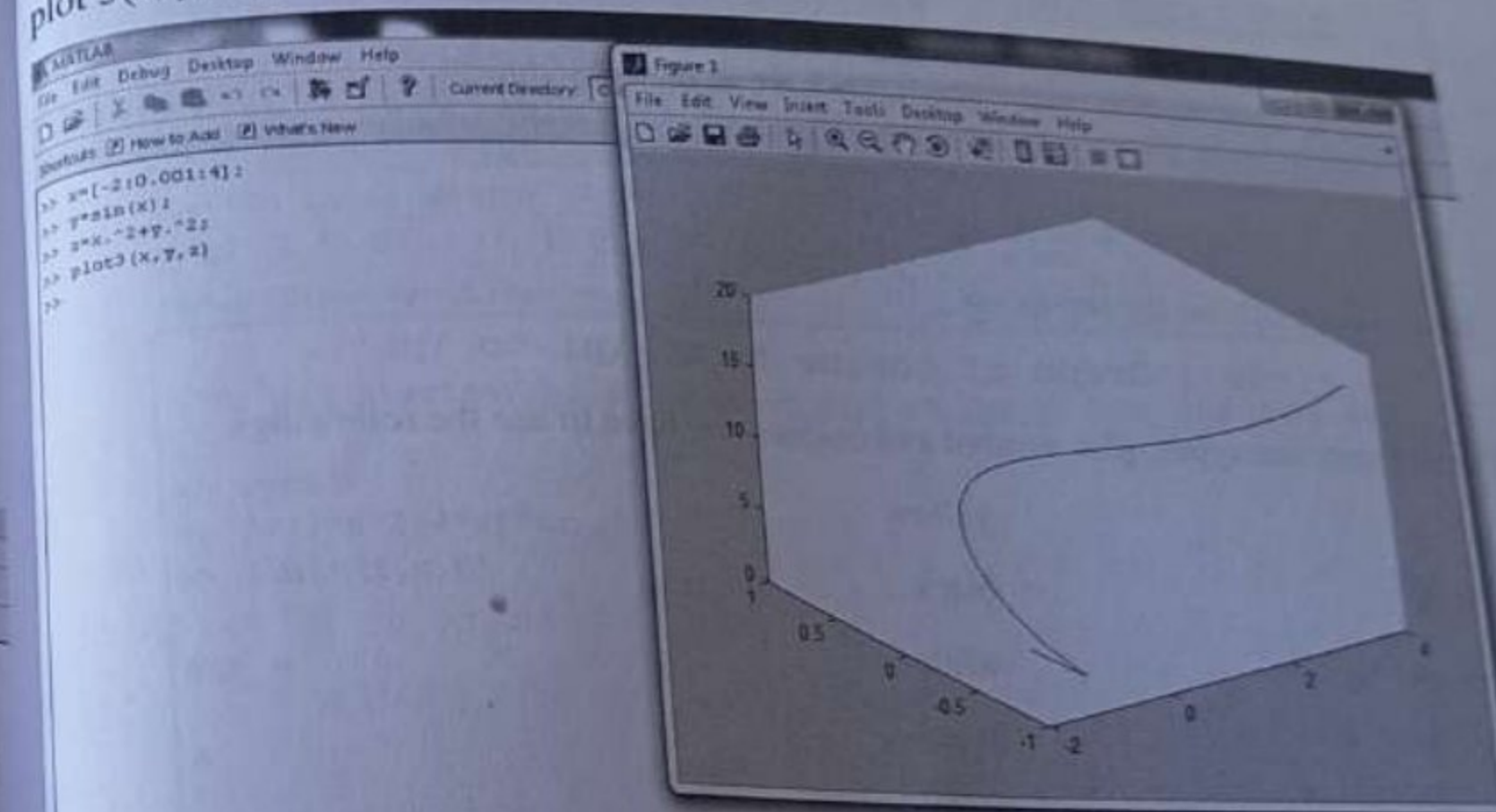
Example:

```
x = [-2 : 0.001 : 4];
```

```
y = sin (x);
```

```
z = x.^2 + y.^2;
```

`plot 3(x, y, z).`



Other commonly used commands for 3-D plotting are

`plot 3(x, y, z)`

`mesh (x, y, z)`

`surf (x, y, z)`

`contour (x, y, z)`

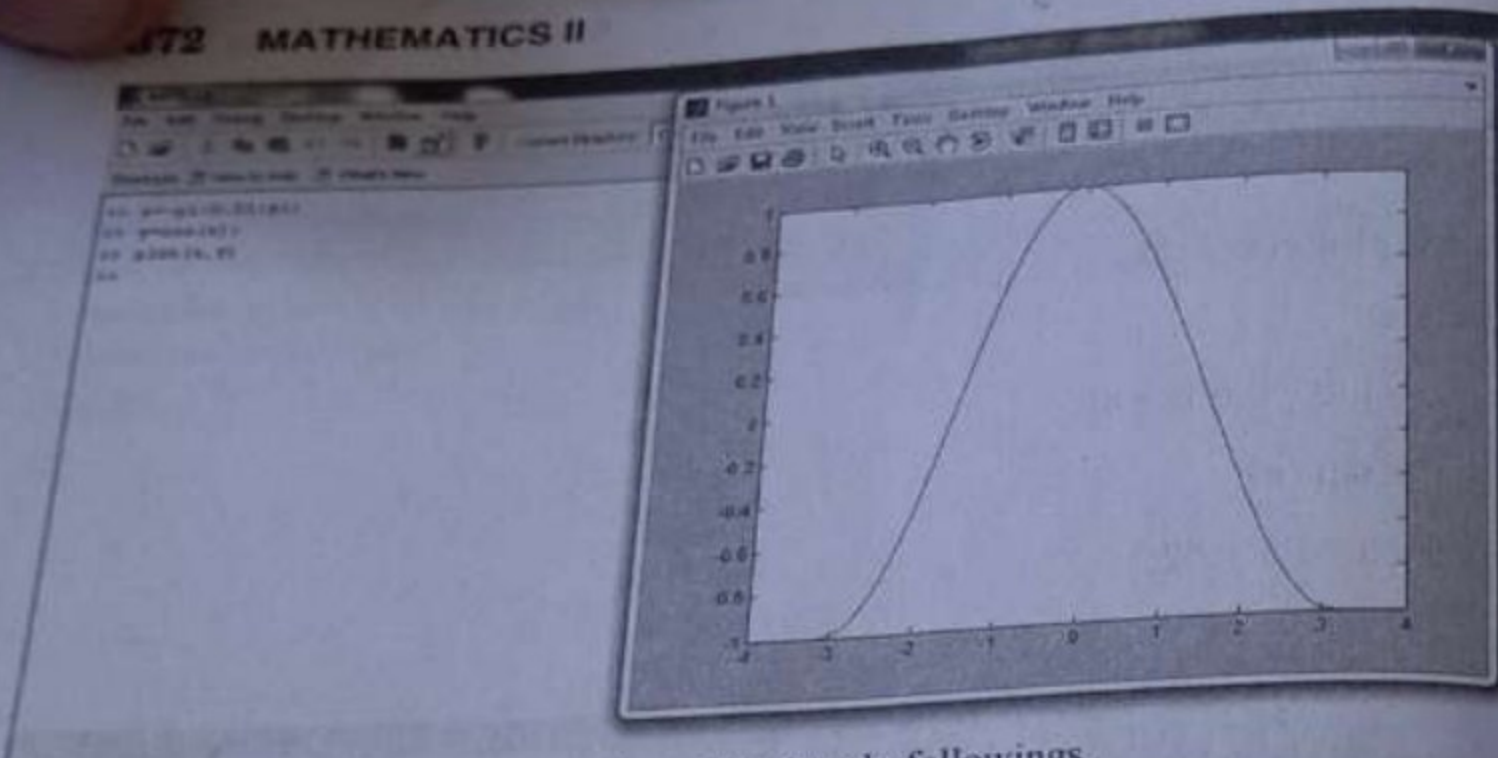
Example

To obtain the graph of $y = \cos (x)$ from $-\pi$ to π , we first define the vector with components equally spaced numbers between $-\pi$ and π , with increment 0.01 (say).

```
>> x = -pi:0.01:pi;
```

```
>> y = cos (x);
```

```
>> plot (x,y)
```

In order to label the axis on the graph, we have to do followings.

```
>> xlabel('x')
>> ylabel('y = cos(x)')
```

We can put a title on the top using

```
>> title('Graph of cosine from -\pi to \pi').
```

For different line types, plot symbol and colours, we have to use the followings.

y	yellow
k	black
w	white
g	green
b	blue
r	red
.	point
*	star
-	solid
--	dashed

For example, to obtain the graph in red, we type

```
>> plot(x, y, 'r')
```

Command	Explanation
who	Gives a list of the variables in use
whos	Gives a list of the variables in use as well as some extra information
clear	Removes all variables
clc	Clear all
clear x y	Removes the variables x and y

MATLAB commands for limit, derivative and integration

Mathematical operation	MATLAB command
1. $\lim_{x \rightarrow a} f(x)$	limit (f, x, a) or limit (f, a)
2. $\lim_{x \rightarrow a^-} f(x)$	limit (f, x, a, 'left')
3. $\lim_{x \rightarrow a^+} f(x)$	lim (f, x, a, 'right')
4. $\frac{d}{dx}(y)$	diff (y)
5. $\int y dx$	int (y)

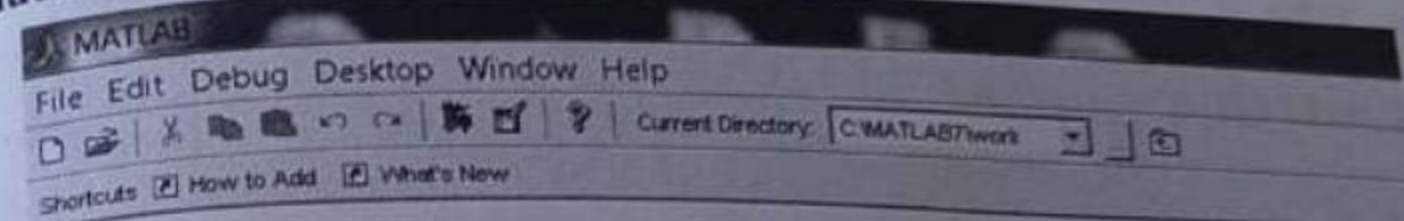
Examples:

Using MATLAB, evaluate.

1. $\lim_{x \rightarrow 0} \frac{7x^2 + 4x}{x}$

2. $\lim_{x \rightarrow y} \frac{\cos x + \cos y}{x - y}$

Solution



To get started, select **MATLAB Help** or **Demos** from the Help menu.

```
>> syms x
>> f = (7*x^2 + 4*x) / x;
>> limit(f, x, 0)

ans =

4

>> syms x y
>> f = (cos(x) - cos(y)) / (x - y);
>> limit(f, x, y)

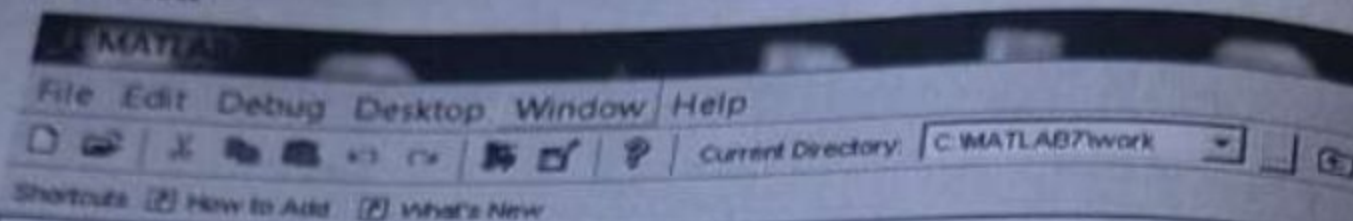
ans =

-sin(y)
```

Using MATLAB, find

1. $\frac{d}{dx}(x^6)$
2. $\frac{d}{dx}(x^2 \tan x)$
3. $\frac{d}{dx}(\log(3x - 2))$

Solution



```
>> syms x
>> y=x^6;
>> diff(y)
```

ans =

6*x^5

```
>> syms x
>> y=x^2*tan(x);
>> diff(y)
```

ans =

2*x*tan(x)+x^2*(1+tan(x)^2)

```
>> syms x
>> y=log(3*x-2);
>> diff(y)
```

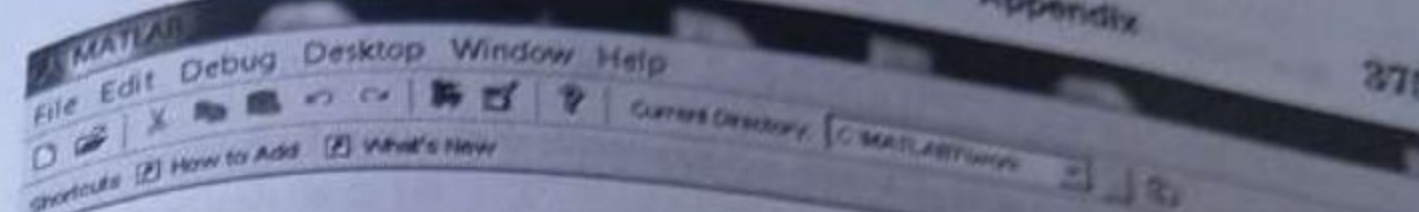
ans =

3/(3*x-2)

Examples:

Using MATLAB, find

1. $\int x^4 dx$
2. $\int \sin x dx$



```
>> syms x
>> y=x^4;
>> int(y)
```

ans =

1/5*x^5

```
>> syms x
>> y=sin(x);
>> int(y)
```

ans =

-cos(x)

Linear Programming Problem

Let f be a column vector of length n and b be a column vector of length m , and let A be a $m \times n$ matrix.

A linear programming associated with f , A and b is the minimum problem

$$\min f^T x$$

or

the maximum problem

$$\max f^T x.$$

subject to $Ax \leq b$

If f is a maximization problem, we change it to the corresponding minimization problem associated with $-f$, A , b .

Solution of LPP minimization problem with MATLAB.

Without equality constraint, the syntax is

$$x = \text{linprog}(f, A, b)$$

To obtain minimum value type $[x, f_{\min}] = \text{linprog}(f, A, b)$

If inequality and equality constraints are given, use

$$x = \text{linprog}(f, A, b, A_{\text{eq}}, b_{\text{eq}})$$

For example:

$$\max f = 7x_1 + 5x_2$$

$$\text{subject to } x_1 + 2x_2 \leq 6, 4x_1 + 3x_2 \leq 6, x_1 \geq 0, x_2 \geq 0.$$

We write this as

$$\min f = -7x_1 - 5x_2$$

$$\text{subject to } x_1 + 2x_2 \leq 6, 4x_1 + 3x_2 \leq 6, -x_1 \leq 0, -x_2 \leq 0.$$

$$\text{Then } \max f = -\min f$$

We solve this in MATLAB in the following way

```
>> f = [-7; -5];
>> b = [6; 6; 0; 0];
>> A = [1 2; 4 3; -1 0; 0 -1];
>> [x, f_min] = linprog(f, A, b)
```

The window will display

1.5000

0.0000

$f_{\min} =$

- 10.5000

Hence $f_{\max} = 10.5$ at $x_1 = 1.5$ and $x_2 = 0$.

MATLAB for Bisection method

Using bisection method, find the root of the equation $x^3 - 2x - 5 = 0$ lying between 2 and 3 correct to 3 places of decimals with error less than 0.001.

Solution

Editor - Untitled*

File Edit Text Cell Tools Debug Desktop Window Help

Stack Base

```
1 clear all;
2 close all;
3 clc;
4 f=@(x)x^3-2*x-5;
5 a=2;b=3;tol=0.001;
6 while abs(a-b)>=tol
7     x0=(a+b)/2
8     if f(a)*f(x0)<0;
9         b=x0;
10    else
11        a=x0;
12    end
13    root=x0
14 end
```

The MATLAB will display the solution when you run this file.

MATLAB for Newton-Raphson method

Using Newton-Raphson method, find a root of $2x^2 - 3x - 1 = 0$ taking $x_0 = 1$ with error less than 10^{-4} .

Solution

MATLAB

File Edit Debug Desktop Window Help

Current Directory: C:\MATLAB7\work

To get started, select MATLAB Help or Demos from the Help menu.

```
>> f=@(x)2*x^2-3*x-1;
>> Df=@(x)4*x-3;
>> x0=1;tol=0.0001;diff=1;
>> while diff>=tol
>> x1=x0-f(x0)/Df(x0);
>> diff=abs(x1-x0);
>> x0=x1;
end
>> root=x0
```

root =

1.7808

Matrix Equation

Let $Ax = b$ be a system of equation in matrix form.

Then the solution is

$$x = A \setminus b.$$

For example:

Solve

$$x + y + z = 9; 2x + 5y + 7z = 52, 2x + y - z = 0$$

In MATLAB,

```
>> A = [1 1 1; 2 5 7; 2 1 -1];
>> b = [9 52 0]';
>> X = A \ b
```

Enter

x =

1

3

5

MATLAB code for Simpson's $\frac{1}{3}$ rule

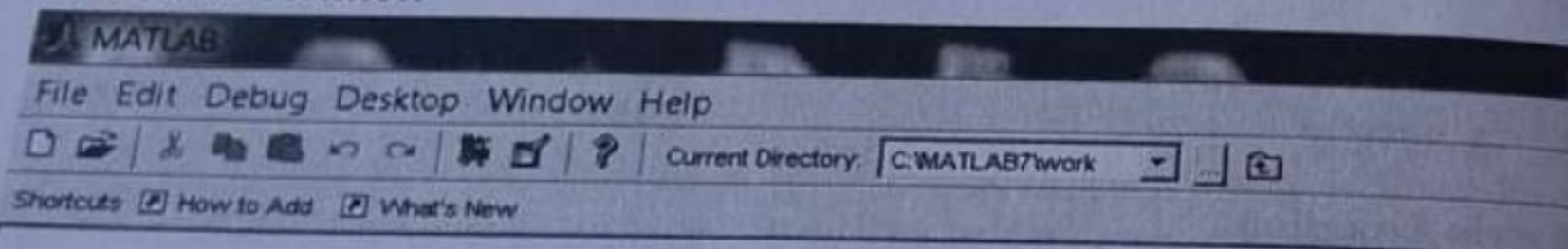
```

f = @(x) (function of x);
a = lower limit;
b = upper limit;
n = multiplies of two;
h = (b - a) / n;
s = f(a) + f(b);
for i = 1 : n - 1
    s = s + 4 * f(a + i * h);
end
for k = 2 : 2 : n - 2;
    s = s - 2 * f(a + k * h);
end
I = (h / 3) * s.

```

For example: $\int_0^1 \frac{dx}{1+x^2}, n=4.$

MATLAB solution



To get started, select MATLAB Help or Demos from the Help menu.

```

>> f=@(x) (1/(1+x^2));
a=0;
b=1;
n=4;
h=(b-a)/n;
s=f(a)+f(b);
for i=1:n-1
    s=s+4*f(a+i*h);
end
for k=2:2:n-2;
    s=s-2*f(a+k*h);
end
I=(h/3)*s
I =
0.7854

```

MATLAB code for Trapezoidal rule

```

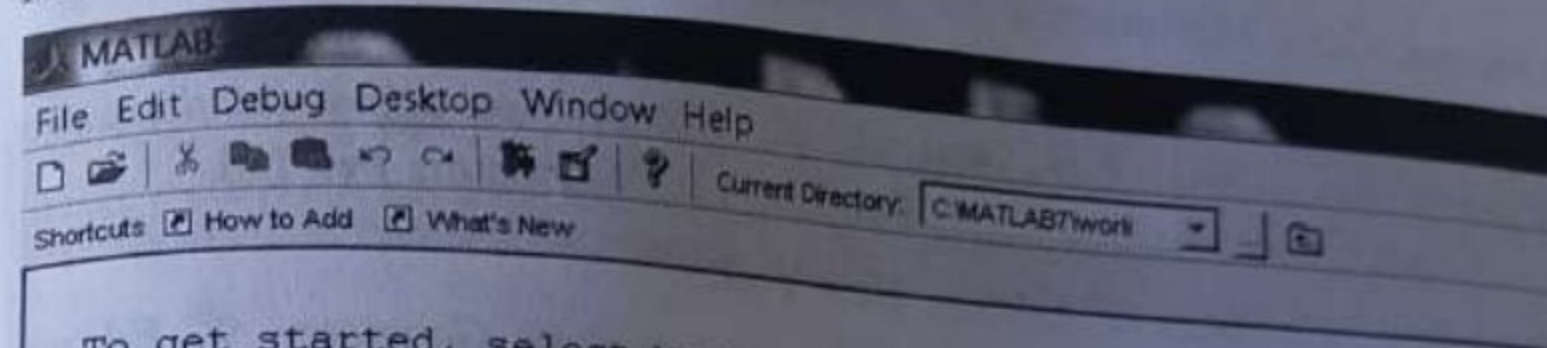
f = @(x) (function of x);
a = lower limit;
b = upper limit;
n = number of subintervals;
h = (b - a) / n;
s = 0.5 * (f(a) + f(b));
for i = 1 : n - 1
    s = s + f(a + i * h);
end
I = h * s.

```

For example:

$$\int_0^{\pi} \sin x \, dx, n=4.$$

MATLAB solution



To get started, select MATLAB Help or Demos from the Help menu.

```

>> f=@(x) sin(x);
a=0;
b=pi;
n=4;
h=(b-a)/n;
s=0.5*(f(a)+f(b));
for i=1:n-1
    s=s+f(a+i*h);
end
I=h*s
I =
1.8961

```


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